Time-Varying Rates of Return (+ Bonds + Yield Curve)

In this chapter, we maintain the assumptions of the previous chapter:

- We assume perfect markets, so we assume four market features:
  1. No differences in opinion.
  2. No taxes.
  3. No transaction costs.
  4. No big sellers/buyers—we have infinitely many clones that can buy or sell.
- We assume perfect certainty, so we know what the rates of return on every project are.

- **But we no longer assume equal rates of returns in each period (year)!**
- Oranges cost more in the winter than in the summer. Why can projects not have different prices (rates of return) at different times, either?

Time-Varying Rates of Returns

Important: All earlier formulas hold.

- The only difference is that $(1 + r_{0,t}) \neq (1 + r)^t$.
- The main complication is that we now need many subscripts—one for each period. For example

$$(1 + r_{0,3}) = (1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3})$$

$$NPV = C_0 \; + \; \frac{C_1}{(1+r_{0,1})} \; + \; \frac{C_2}{(1+r_{0,2})} \; + \; \frac{C_3}{(1+r_{0,3})}$$

$$= C_0 \; + \; \frac{C_1}{(1+r_{0,1})} \; + \; \frac{C_2}{(1+r_{0,1})\cdot(1+r_{1,2})} \; + \; \frac{C_3}{(1+r_{0,1})\cdot(1+r_{1,2})\cdot(1+r_{2,3})}$$

If you like it more formal,

$$(1 + r_{t,t+i}) = (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) \cdots (1 + r_{t+i-1,t+i})$$

$$= (1 + r_{t+1}) \cdot (1 + r_{t+2}) \cdots (1 + r_{t+i}) = \prod_{j=t+1}^{t+i} (1 + r_j)$$

$$\text{PV} = \sum_{t=1}^{\infty} \left(\frac{C_t}{1+r_{0,t}} \right) = \sum_{t=1}^{\infty} \left[\frac{C_t}{\prod_{j=1}^{t}(1+r_j)} \right]$$

Recall that r_j is an abbrev for $r_{j-1,j}$.

Here is a computer program that executes this formula.
It relies on two subroutines, cashflow(time) and discountrate(timestart, timeend).

```
discountfactor ← 1.0;
npv ← 0.0;
for time=0 to infinity do
    begin
        discountfactor ← discountfactor/(1 + discountrate(time − 1, time))
        npv ← npv + cashflow(time) ∗ discountfactor;
    end
return npv;
```

Annualized Rates of Returns

Q1: Is 1,573 miles in 28.6 hours fast or slow?

In one sense, speed is a good measure by which we can compare runners as to their rates of accumulation of distance per unit of time. So, we can measure sprinters, marathon runners, cars, planes, etc. In another sense, sprinters cannot be compared to marathon runners. Speeds are necessarily different. 15mph over 100m is not necessarily better or worse than 10mph over 10 miles—but 15mph is a faster rate than 10mph.

The same applies to interest rates. We need a standardized form of "rate of accumulation" by which we can compare, e.g., 3 day interest rates, with 5 year interest rates.

Q2: Your project will give you a rate of return of 100% (double your money) over 15 years. Is this a lot or a little?

Q3: How does this rate of capital accumulation compare to 1% over 3 months?

Q4: If the 1-year interest in year 1 is 5%, and the 1-year interest rate in year 2 will be 3%, what is the annualized interest rate?

Q5: Is an annualized interest rate more like an average or more like a sum?

Important: Almost all interest rates are quoted as annualized.

Annualized interest rates are (often just a little) below average interest rates, because they take into account the interest on interest.

Inflation: Real and Nominal Rates

- A nominal cash flow is simply the number of dollars you pay out or receive.
- A real cash flow is adjusted for inflation. A real dollar always has the same purchasing power.
- If the U.S. were to call everything that is a cent today a dollar henceforth, inflation would be 9,900%—and yet it would not matter as long as the contracts today are clear about the units (dollars) and their translation.

  If properly contracted for, inflation is not a market imperfection.

  Just because quoted prices are less in Euros than in Lira can be called deflation, but it does not in itself create a problem.

  (If you need it even clearer, realize that a Euro is not the same as a Lira. In the same way, a Euro next year is not the same as a Euro this year.)
- In sum, inflation per se is not a friction (or market imperfection)—if everything is contracted in real terms. However, in the real world, most contracts are in nominal terms, so as an investor you must worry about inflation.

How do Nominal and Real Rates Relate?

- You have \$100, which you invest for 1 year at 10%.
- Bread sells for \$2.00 today.
- Your \$100 can purchase ____ loaves today.
- *Bread Inflation* over the next year will be 4%.
- The bank pays a nominal rate of return of 10% per year.
- Next year, bank will pay you ____ *nominal* dollars.
- Next year, one loaf of bread will cost ____.
- Thus, if you put your money in the bank and earn the nominal interest rate, you will be able to purchase ____ ____ loaves of bread.
- In real terms, you would start with ____ loaves of bread, and earn an additional ____ loaves of bread.
- Thus, in real terms your rate of return is ____

<u>Repeat:</u> With an inflation rate of 4% and a nominal rate of return of 10%, in real dollars you begin with ____, earn a real rate of return of ____. From the latter, in terms of purchasing power today, you finish not with \$110, but with ____ *real* dollars.

Q6: Can you relate the three rates to one another?

The Formula

More generally:

$$\begin{array}{ccccc} (1+0.0577) & \cdot & (1+0.04) & \approx & (1+0.10) \\ (1+\text{real rate}) & \cdot & (1+\text{inflation rate}) & = & (1+\text{nominal rate}) \end{array} .$$

Important: You must remember this formula!

Intuition: Why is this a "one-plus" type formula? Sorry, my intuition is not that good. I convince myself with examples here.
When all rates are very small, the approximation

$$\text{real rate} + \text{inflation rate} \approx \text{nominal rate}$$

can be acceptable, *depending on the circumstances*, but this approximation formula is *not* exactly correct.

- One real dollar today equals one nominal dollar today. (Usually!)
- An inflation-adjusted dollar is $\$1/(1+\pi)$. So, $110 next year is $\$110/1.04 \approx \$105.77$ today in inflation-adjusted dollars. $100 nominal next year is $96.15 real dollars today. Etc.
- Sometimes, real dollars are also called "inflation-adjusted" dollars, or—and this is where it gets real awful—are even called "in today's dollars." Unfortunately, different people mean difference things by these phrases. *In case of doubt, ask!!*

Inflation in NPV

- .
  Q7: A project will return \$110 in nominal cash next year. The cost of capital is 10%. What is the PV?

- .
  Q8: The inflation rate is 4%. A project will return \$110 in nominal cash next year. What is the purchasing power of the future \$110 in today's *real* dollars?

- .
  Q9: The inflation rate is 4%. The cost of capital is 10%. What is the *real* cost of capital?

- .
  Q10: What is the project's *real* dollar value discounted by the *real* cost of capital? Why?

Conclusion

Important: You can either discount nominal dollars with nominal interest rates, or real dollars with real interest rates. Never mix.

What is the current inflation situation?

Q11: What is today's interest rate?

Q12: What is the inflation rate today?

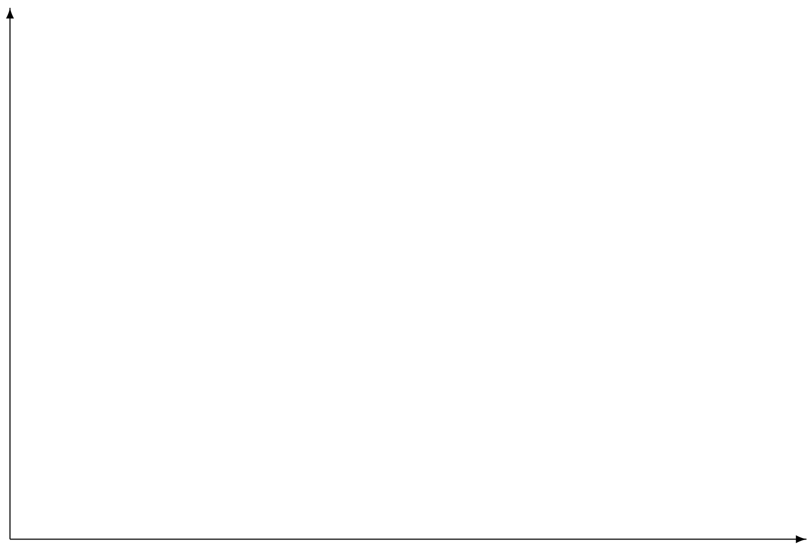
The Yield Curve and Treasuries

- U.S. Treasuries are one of the most important financials markets in the world. (Only the mortgage bond market may be bigger, by some accounts.)
  - They are risk-free.
  - The outstanding amount in 2008 was >$9.5 trillion.
    (second-largest market, after mortgage securities.)
  - Annual trading is ≈$80-$100 trillion. (Turnover = 10 Times!)
  - Names: Bills (–0.99y), Notes (1y–10y), Bonds (10y–).

- This market is close to "perfect":
  - Extremely low transaction costs (for traders).
  - Few opinion differences (inside information).
  - Deep market—many buyers and sellers.
  - Income taxes depend on owner.
- In addition, there is no uncertainty about payment. (However, a market could still be perfect, even if payoffs are uncertain.)
- In many ways, (zero coupon) Treasuries are the simplest possible financial instrument in the world.

Yield Curves: Sample Shapes

A yield curve is the plot of annualized yields (Y-axis) against time-to-maturity.



Q13: Can the Treasury yield curve be flat? Can it slope down? What does this mean?

IMPORTANT: **The yield curve is a fundamental tool of finance. It always graphs annualized rates. It measures differences in the costs of capital for (risk-free) projects with different horizons.**

In the real world, many variations on the yield curve are in use, e.g., yield-curves constructed from risky bonds or from foreign bonds.

Other Yield Curve Factoids

Q14: Is the 3-year bond a better deal than the 1-year bond?

Q15: What is the most common yield curve shape?

Q16: What does an upward sloping or downward sloping yield curve mean for the economy (not for an investor)?

Q17: How does today's inflation rate compare to today's interest rate?

Q18: ...after personal income taxes?

Q19: Does the Fed control the yieldcurve?

Names: Spot and Forward Rates

We call a currently prevailing interest rate for an investment starting today a *spot* interest rate. Like all other interest rates, spot rates are usually quoted in annualized terms.

Q20: What is the annualized spot rate on a 1-month U.S. T-bill today?

Q21: What is the annualized spot rate on a 30-year U.S. T-bond today?

A forward rate is an interest rate that will be applicable in the future. It is the opposite of a spot rate.

Our main questions now:

Q22: What does the yield curve today imply about future interest rates?

Q23: Can you lock in future interest rates today?

More Painful Notation

- We denote an annualized interest rate over 15 years as $r_{\overline{15}}$. This contrasts with our notation for the 15-year non-annualized holding interest rate, which is $r_{0,15}$.

$$
\begin{aligned}
(1+r_{\overline{15}})^{15} &\equiv (1+r_{0,15}) \equiv (1+r_{0,1})\cdot(1+r_{1,2})\cdot\ldots\cdot(1+r_{14,15})\\
\Longleftrightarrow\quad (1+r_{\overline{15}}) &\equiv (1+r_{0,15})^{1/15}\\
(1+r_{\overline{t}})^{t} &\equiv (1+r_{0,t})\\
\Longleftrightarrow\quad (1+r_{\overline{t}}) &\equiv (1+r_{0,t})^{1/t}
\end{aligned}
$$

<u>Example:</u> $r_{0,5} = 27.63\% \iff r_{\overline{5}} = 5\%$.

This is our notation, and not necessarily used elsewhere. To make matters worse, some people will use R to mean $1+r$, believing you can figure out whatever they may have meant. Others will just capitalize R and mean the same thing, namely r. Sigh...

- Notation Summary:

$$
\begin{aligned}
(1+r_{0,1}) &= (1+r_{\overline{1}})^{1} = (1+r_{0,1})\\
(1+r_{0,2}) &= (1+r_{\overline{2}})^{2} = (1+r_{0,1})\cdot(1+r_{1,2})\\
(1+r_{0,3}) &= (1+r_{\overline{3}})^{3} = (1+r_{0,1})\cdot(1+r_{1,2})\cdot(1+r_{2,3})
\end{aligned}
$$

- The interest rate from period 1 to period 2 is called the *1-Year Forward (Interest) Rate from Period 1 to Period 2*.
- In a world of certainty, the forward rate will be the future spot rate: We know it! (Later I will show you how you can make the forward rate your personal future spot rate, even in a world of uncertainty.)

Approximate Answers

Remember:

- An annualized rate of return is more like an average.
- A holding rate of return is more like the sum.

A 1-year bond has an (annual) rate of return of 5%. When the first bond will come due, you will be able to purchase another 1-year bond that will have an (annual) rate of return of 10%. When the second bond will come due, you will be able to purchase another 1-year bond that will have an (annual) rate of return of 15%.

Calculator VERBOTEN. Use only your intuition.

Rates of Return

Spot + Forward	Holding	Annualized
$r_{0,1}$ =5%	$r_{0,1}$ =	$r_{\overline{1}}$ =
$r_{1,2}$ = 10%	$r_{0,2}$ =	$r_{\overline{2}}$ =
$r_{2,3}$ = 15%	$r_{0,3}$ =	$r_{\overline{3}}$ =

Exact answers will be calculated next.

Q24: Can you guess whether numbers will be a little lower or higher?

A Set of Consecutive 1-Year Bonds

Q25: W/o a calculator, what are the 3 holding rates of return, roughly?

Q26: What are they exactly? (Name and Notation)

Q27: What are the 3 annualized rates, roughly? (Name and Notation)

Q28: What are the 3 annualized interest rates exactly?)

A Set of Annualized Rates on Longer and Longer Bonds

5-4,3-8

A 1-year bond has an annualized rate of return of 5% per year. A 2-year bond has an annualized rate of return of 10% per year. A 3-year bond has an annualized rate of return of 15% per year.

Q29: W/o a calculator, what are the three (total) holding rates of return?

Q30: What are they exactly? (Name and Notation)

Q31: W/o a calculator, what are the 3 annual rates of return? (**Name** and Notation)

Q32: What are they exactly?

Holding	Annualized	Spot/Forward
$r_{0,1} =$	$r_{\overline{1}} =$	$r_{0,1} =$
$r_{0,2} =$	$r_{\overline{2}} =$	$r_{1,2} =$
$r_{0,3} =$	$r_{\overline{3}} =$	$r_{2,3} =$

IMPORTANT: When you work with the yieldcurve, use your over-the-envelope intuition to know what the order of magnitude of your answer should be.

Because the annualized yield is an average of spot/forward rates, the forward rates rises/declines faster than the yield curve. For example, if $r_{\overline{1}} = 5\%$ and $r_{\overline{2}} = 6\%$, then $r_{1,2} > 6\%$, because 5% and $r_{1,2}$ "geo-averaged" must come to 6%. By this argument, $r_{1,2}$ should be about 7%.

Yield Curve Concepts

Q33: Given a full set of annualized interest rates, $r_{\bar{1}}$, $r_{\bar{2}}$, ..., $r_{\bar{T}}$, can you compute each and every T-year holding rates of return?

Q34: Given a full set of annualized interest rates, $r_{\bar{1}}$, $r_{\bar{2}}$, ..., $r_{\bar{T}}$, can you compute each and every forward rate (implied future spot rate) from $T-1$ years to T years?

Q35: Does the "yield curve" imply a unique set of forward/spot rates?

Q36: Does the complete set of spot holding rates of returns imply a unique yield curve?

Q37: Does the complete set of spot prices on zero (or other) bonds imply a unique yield curve?

Q38: Does the complete set of forward rates (plus 1-year spot rate) imply a unique yield curve?

Yield Curves

Q39: Can you plot the yield curve, given a complete set of interest rates (of any type)?

IMPORTANT: **The "yield curve" or "term structure of interest rates" is the curve plotting the spot (i.e., annualized) interest rate on the y-axis against the time of the payment on the x-axis. It implies all forward interest rates.**

Look up Today's Yield Curve in the WSJ.

Nerd note: Although we pretend that the WSJ quotes true 2-year interest rates, it actually quotes interest rates from 2-year coupon bonds. We know that the duration for such bonds is shorter than the maturity. Usually, the difference is not big. Unless you are a bond trader, this difference can typically be ignored.

Q40: What does an upward-sloping yieldcurve mean for an investor?

Q41: What could a "risk premium" do to a yield curve in which risk is higher for longer-term investments?

Get Rich From Longer-Term Bonds?

The 1-year bond earns an annualized 5%, the 2-year bond earns an annualized 10%, 3-year bond earns an annualized 15%. Is the 1-year a worse deal than the 3-year, even if you want to sell in one year?

Q42: What does the 3-year zero bond cost today?

It costs $\$1,000/1.15^3 \approx \$657.52$ per $1,000 face value today.

Q43: What are the forward interest rates (and, in our certain world, the future interest rates)?

$r_{2,3} = 1.15^3/1.10^2 - 1 \approx 25.69\%$ and $r_{1,2} = 1.10^2/1.05^1 - 1 \approx 15.24\%$.

Q44: What will the 3-year zero bond cost in one year?

$$\$1,000/(1.1524 \cdot 1.2569) \approx \$690.45$$

Q45: What holding rate of return does the 3-year zero bond give you from time 0 to time 1?

$$r \approx \$690.45/\$657.52 - 1 \approx 5\%$$

This is the same return as you get from purchasing the 1-year bond.

Q46: What rate of return will the 3-year zero bond offer from time 1 to time 3?

The forward rate, of course.

Yield Curve Changes

Q47: What happens to the value of a bond (a loan) that you already own when interest rates increase? Does loan length matter?

Here is an example of a bond promising 8%/year:

- A 30 year bond that promises 8% interest rate costs ($\$100/1.08^{30} \approx$) \$9.94 for each \$100 promise in payment.
- If the interest rate increases by 10 basis points, the price changes to \$9.67.
- The holding rate of return is $\$9.67/\$9.94 - 1 \approx -2.74\%$. For each \$100 in investment, you would have just lost \$2.74!
- For a 1-year bond, the same calculation $p_0 = \$100/1.08 \approx \$92.5926$, $p_1 = \$100/1.081 \approx \$92.507$, and $r = p_1/p_0 - 1 \approx -0.09\%$.
- For a 1-day bond, the calculation $p_0 = \$100/1.08^{1/365} \approx \$99.979$, $p_1 = \$100/1.081^{1/365} \approx \$99.9787$, and $r = p_1/p_0 - 1 \approx -0.00025\%$. In fact, a 1-day bond is practically risk-free.

Conclusion: The interest rate sensitivity of a 30-year bond is much higher than that of a 1-year (or 1-day bond).

Q48: Is a 1-day bond riskier or a 30-year bond?

If we allow for uncertainty, long-term bond investors can get more return for two reasons: because of higher expected rates of returns in the future [e.g. due higher future inflation rates], or because they are earning a "risk premium" (to be discussed soon). The evidence suggests it is more of a risk premium than expectations of higher future rates.

Corporate Lesson

IMPORTANT:

- **A project of x-years is not simply the same as investing in x consecutive 1-year projects. From an investment perspective, they are different animals, and can require different costs of capital.**
- **The fact that longer-term projects may have to offer higher rates of return (could but) need not be due to higher risk. Even default-free Treasury bond projects in the economy that are longer-term have to offer higher rates of return than default-free Treasury bond projects in the economy that are shorter term.**
- **(Of course, long-term projects are also often riskier (more default), and this may eventually also help explain why long-term projects have to offer higher rates of return.)**

Appendices (Omitted)

(The appendix will be part of the exams.)

Locking Forward Rates (5-A.b.): Given the current yield curve, you can lock in the future interest rate today. That is, you can eliminate all uncertainty about what interest rate that you will have to pay (or that you can earn). For example, you can buy and short Treasuries to lock in a 1-year saving Treasury rate for $1 million beginning in year 3 and lasting until year 4.

Future Interest Rates vs. Forward Rates: In the real world, future interest rates can be different from forward rates. Indeed, if you lock in a, say, 10-year-ahead 1-year savings interest rate today, on average you would have earned a higher rate of return than you would have if you had purchased 1-year savings bonds in the open market. If you are dealing with bonds, you therefore may need more notation. You now will have a future 1-year spot rate in 2030 (say $r_{2030,2031}$), and a 1-year forward rate that you can lock in today (say $f_{\text{Now},2030,2031}$, which is the 1-year forward rate locked in today. Tomorrow's locked in forward rate would be $f_{\text{tomorrow},2030,2031}$. And so on. Yikes.

Duration (5-A.c-e): A project that pays $100 in one year and $100 in two years has a maturity of two years, the same as a "zero-" project that pays only $200 in two years. However, the first project is clearly shorter-term. Duration is a measure of when the cash flow arrives "on average." It is in common use in the bond context, but useful for all sorts of projects. It is also often used for "hedging"—matching projects to be similar.

Continuous Compounding (5-A.f): If interest is paid not once per year, but every second, this is the continuously compounded interest rate. It is often used for options pricing. OK, skipped for exams.

Strips (5-A.g): I cheated on the exact method to compute bond prices. The common yieldcurve is computed from IRRs, and not even based on actual interest rates, but based on interest quotes.

Homework Assignment

1. Reread Chapter 5.
2. Read Chapter 6.
3. Hand in all Chapter 5 end-of-chapter problems, due in 7 days.