Perpetuities and Annuities

In this chapter, we maintain the assumptions of the previous chapter:

- We assume perfect markets, so we assume four market features:
  1. No differences in opinion.
  2. No taxes.
  3. No transaction costs.
  4. No big sellers/buyers—we have infinitely many clones that can buy or sell.
- We assume perfect certainty, so we know what the rates of return on every project are.
- We assume equal rates of returns in each period (year).

Important: You need your calculator! Have chapter 3 read.

Questions

- Are there any shortcut NPV formulas for long-term projects—at least under certain common assumptions?
- Or, do we always have to compute long summations for projects with many, many periods?
- Why do some of the folks in the room have the ability to quickly tell you numbers that would take you hours to figure out?
- How are loan payments (e.g., for mortgages) computed?

- If your firm produces $5 million/year forever, and the interest rate is a constant 5% forever, what is the value of your firm?
- If your firm produces $5 million/year in *real* (inflation-adjusted) terms forever, and the interest rate is a constant 5% forever, what is the value of your firm?
- What is the value of a firm that generates $1 million in earnings per year and grows by the inflation rate?
- What is the monthly payment on a 6% 30-year fixed rate mortgage?
- NPV and Excel are a pain. Can't you teach us any shortcuts so that we can do the calculations in our heads as fast as the "quants" in our meeting?

  You can think of perpetuities and annuities as shortcut formulas that can make computations a lot faster, and whose relative simplicity can sometimes aid intuition.

Simple Perpetuities

A perpetuity is a financial instrument that pays C dollars per period, forever. If the interest rate is constant and the first payment from the perpetuity arrives in period 1, then the PV of the perpetuity is:

$$(\text{PV} =) \qquad \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

Summation notation is *very common* in finance. It makes it easier if you are comfortable with its meaning! It is just notation, not really a new concept. More explanation: t is not an input variable; only C and r are. t is part of the notation that helps tell you how many terms you have

IMPORTANT: Make sure you know when the first cash flow begins (tomorrow [$t = 1$], not today [$t = 0$]!).

I sometimes write C_{+1}/r to remind myself of timing, even though cash flows are the same at time 1 as they are at time 25—I could have written C_{25} instead.

Q1: Write out what the perpetuity formula means!

If you know how to program, the summation $\sum_{i=1}^{100} f(i)$ is the same as

sum $\Leftarrow 0.0$;
for **i** *from* 1 *to* 100 *do begin* **sum** $\Leftarrow$ **sum** $+ f($**i**$)$ *end*
return **sum** ;

Note that **i** is not an input variable—instead, it is a device to indicate that we have 100 terms which we want to sum up. Aside, IMHO, programming teaches logical thinking. Aside, you need to know how to do basic programming in almost any job these days. **IMHO, you should learn basic programming asap!**

Q2: What is the value of a promise to receive $10 forever, beginning next year, if the interest rate is 5% per year?

Q3: What is the value of a promise to receive $10 forever, beginning this year, if the interest rate is 5% per year?

Q4: What is the value if the first cash flow starts today rather than tomorrow?

Omitted Nerd Note: Time Consistency

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Is the formula time-consistent? For example, if my house/property is paying up $100 eternally, and I get cash, how can it still be worth the same tomorrow as it is today?

Question is: if you have a perpetuity worth $1,000, you will still have an annuity worth $1,000 next year *and* get one payment, too. How can this be?

Q5: Next year, you will still have a perpetuity (then beginning the year thereafter, i.e., year 2). How much will the perpetuity be worth next year?

Growing Perpetuities

A growing perpetuity pays C, then $C \cdot (1+g)$, then $C \cdot (1+g)^2$, then ... For example, if $C = \$100$ and $g = 0.10 = 10\%$, then you will receive the following payments:

$C_0 = 0$	= $0	(no discount)
$C_1 = \$100$	= $100.00	(then discount with $r_{0,1}$)
$C_2 = \$100 \cdot (1+10\%)$	= $110.00	(then discount with $r_{0,2}$)
$C_3 = \$100 \cdot (1+10\%)^2$	= $121.00	(then discount with $r_{0,3}$)
$C_4 = \$100 \cdot (1+10\%)^3$	= $133.10	(then discount with $r_{0,4}$)
$C_5 = \$100 \cdot (1+10\%)^4$	= $146.41	(then discount with $r_{0,5}$)

and so on, forever

The PV of a growing perpetuity can be quickly computed as

$$\text{PV} = \sum_{t=1}^{\infty} \frac{C_1 \cdot (1+g)^{t-1}}{(1+r)^t} = \frac{C_1}{r-g}$$

Important: You must memorize the RHS, and know what it means!

Notice:

- The growth term acts like a reduction in the interest rate.
- The time subscript for the payment matters now, because $C_1 \neq C_2$.

Q6: Check the growing perpetuity formula by hand (well, Excel)!

Q7: What is the value of a promise to receive \$10 next year, growing by 2% (just the inflation rate) forever, if the interest rate is 6% per year?

Q8: What is the value of a firm that just paid \$10 **this** year, growing by 2% forever, if the interest rate is 5% per year?

Q9: What is the formula for the value of a firm which will only grow at the inflation rate, and which will have \$1 million of earnings next year?

Q10: In 10 years, a firm will have annual cash flows of \$100 million. Thereafter, its cash flows will grow at the inflation rate of 3%. If the applicable interest rate is 8%, estimate its value if you will sell the firm in 10 years? What would this be worth today?

Growing perpetuity shortcuts are commonly used, and in many contexts. For example, in "pro-formas," growing perpetuities are typically used to guestimate the present value of the residual firm value after an arbitrary T years in the future. A common long-run growth rate in this formula is then often the inflation rate. (The first T years are computed in more detail.) Typical T's are 10 to 20 years.

The Gordon Dividend Growth Model

Q11: What should be the share price of a firm that pays dividends of \$1/year, whose dividends have grown by 4% every year and will continue to do so forever, if its cost of capital is 12% per annum?

Q12: What is the cost of capital for a firm that pays a dividend yield of 5% per annum today, if its dividends are expected to grow at a rate of 3% per annum forever?

Phrased differently, this is the expected rate of return embedded in the price of the firm today. A higher price would imply a lower cost of capital at which the firms can obtain capital from investors.

Important: Don't trust the GDGM: Dividends are very unstable.

In fact, there is a fairly strong irrelevance proposition here. Given its underlying projects, it should not matter whether the firm pays out \$1 or \$10 in dividends. What it does not pay out in dividends today will make more hey next year. Thus, expected rates of returns obtained from the Gordon model are highly suspect.

A slighly more intelligent application, although without a name, would use earnings instead of dividends. (There can be a similar irrelevance proposition for earnings as there is for dividends [firms can move earnings across periods], but it is not as strong.)

Q13: In 2000, the P/E ratio of the stock market reached about 45. If you assume that these corporations will grow roughly at the overall economy's (GDP) growth rate of 4–5% per year, what should investors have reasonably expected in terms of a likely future rate of return implied by the stock market's level?

Annuities

An annuity is a financial instrument that pays C dollars for T years. It has the following PV formula:

$$\text{PV} = \sum_{t=1}^{T} \frac{C}{(1+r)^t} = \left(\frac{C}{r}\right) \cdot \left[1 - \frac{1}{(1+r)^T}\right]$$

- Make sure you know when the first cash flow begins: It starts tomorrow [$t = 1$], not today [$t = 0$]!
- You should remember this formula, or at least be able to quickly derive it.
  - I confess: when I have not used the formula for a while, I double-check just to make sure.
  - I remember the formula, because an annuity is a perpetuity today, minus a (properly discounted) perpetuity in the future.

    $$\text{PV} = \left(\frac{C}{r}\right) - \frac{1}{(1+r)^T} \times \left(\frac{C}{r}\right)$$

    This is how I remember how the formula must look like.
  - Sometimes, if there are fewer than 30 terms, I am too lazy to even do this. I just use Excel.

Q14: Compute the value of a 3-decade annuity $100 million each decade, with a decadal (10-year) interest rate of 50%, via the plain NPV formula and via the annuities formula. The first payment occurs in 10 years.

Annuity Example: Mortgage Loan

Here is a summary of how mortgage payments are usually calculated: A 30-year mortgage is an annuity with 360 monthly payments, starting one month from today. Because payments are monthly, we need the monthly interest rate. The monthly rate on a mortgage is always computed as the quoted rate divided by 12. (In other words, like bank interest, your actual annual interest rate on a mortgage is higher than quoted. Lovely, isn't it?) So the monthly interest rate on a 9% mortgage is

$$r_{\text{monthly}} = 0.09/12 = 0.0075 \text{ per month}$$

Q15: To buy a house, you intend to take out a $1,200,000 fixed rate mortgage with 30 years to maturity, 360 equal monthly payments, and a quoted interest rate of 9%. (You could also call this 9%, compounded monthly.)

What will be your monthly mortgage payment?

(What do you know, what do you need?)

PS: do you prefer Excel or the formula?

Omitted: Mortgage Interest

Of the first month's payment, how much is interest and how much is principal? What is the balance remaining on the loan after 3 months? After 10 years? Uncle Sam makes you care about this calculation! In addition, you may be curious where the principal+interest numbers from your annual mortgage statement come from.)

Month 1 The monthly interest rate is 0.75%, so the amount of interest due at the end of the first month is

$$0.0075 \cdot \$1,200,000 = \$9,000.00$$

Because $9,000 of the first payment of $\approx\ \$9,655.47$ goes to paying interest, the remaining $\$9,655.47 - \$9,000 \approx \$655.47$ goes to paying off some of the remaining principal on the loan, so the balance on the loan at the end of one month, after making the first payment, is

$$\$1,200,000 - \$655.55 \approx \$1,199,344.53$$

Month 2 Interest charged during month 2 is

$$\$1,199,344.53 \cdot 0.0075 \approx \$8,995.08$$

So $8,995.08 of month 2's payments goes to paying interest, and the remaining

$$\$9,655.47 - \$8,995.08 \approx \$660.39$$

goes to paying off principal, so the balance remaining on the loan after the month 2 payment is

$$\$1,199,344.53 - \$660.387 \approx \$1,198,684.14$$

Month 3

$$\text{Interest} \approx \$1,198,684.14 \cdot 0.0075 \approx \$8,990.13$$

$$\text{Principal Repayment} \approx \$9,655.47 - \$8,990.13 \approx \$665.34$$

$$\text{Remaining balance} \approx \$1,198,684.14 - \$665.34 \approx \$1,198,018.80$$

Month 120 For year 10, we could continue like this for 120 periods.

A simpler method (clever shortcut) is to remember that the remaining balance always equals the present value of the remaining payments, calculated using the loan's interest rate to do all discounting.

(You can compute the remaining balance in any way whatsoever. You might as well do this the slow way in Excel.)

After 10 years (120 months) there are 360 - 120 = 240 payments remaining. So

$$\text{Remaining Balance}_{\textit{after}\ 120\ \text{months}} = \frac{9{,}655.47}{1.0075} + \frac{9{,}655.47}{(1.0075)^2} + \ldots + \frac{9{,}655.47}{(1.0075)^{240}} \approx \$1,073,156.93$$

(Annual) Rental Equivalents (NPV Equivalents)

Assume interest rate $r = 20\%$.

Time	0	1	2
Project A (or Rent A)	\$20	\$12	
Project B (or Rent B)	\$15	\$15	\$15

If this is rent, which should you contract for?

Q16: Which project is cheaper?

Q17: What is the PV (or cost) of project A?

Q18: What is the "equivalent annual rent" of project A?

Q19: What if you need to use the building only for 2 periods and cannot rent it out beyond?

Q20: Why is this stuff in the annuities chapter?

Of course, rental cost and rental (project) income are mutual flip sides.

Remainder of This Handout

- Covering the remaining slides is dependent on available time.
- The subject matter is very important to anyone dealing with bonds (corporate, Treasury, or other).

Homework Assignment

1. Reread Chapter 3.
2. Read Chapter 4.
3. Hand in all Chapter 3 end-of-chapter problems, due in 7 days.

Level-Coupon Bond

- A coupon bond has interim payments.

  Most bonds are coupon bonds.

- Most corporate bonds are level-coupon bonds—this means that the level of the coupons will remain the same over the life of the bond.

  The most common U.S. bond is an **x% semi-annual level coupon bond**; take the principal, multiply it by x% to obtain the annual coupon payment, divide it by two, and this is what is being paid every six months. **The x% is not the interest rate implicit in the bond!** The designation is used to tell you the payment flow of this bond without much ado.

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  - **Q21:** What are the payments to a 5% semi-annual level coupon bond, $100 million, due in 2.5 years?

- A zero bond has no interim payments.
  - **Q22:** How do you earn interest on a bond that gives you no interest payments?
  - **Q23:** Is the coupon rate of a bond equal to the interest rate?

▪ **Q24:** What is the interest rate on an IBM 3.5% semi-annual coupon bond?

- The book works out an example of level coupon pricing. Tedious but straightforward. Make sure you can do it, too.

More Applications

- What formula should we use?
- What numbers do we know?
- What numbers do we seek?

Q25: An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years and sells for \$806,070. What is the implied interest rate (here called an IRR—more soon) that this insurance company is offering you?

Q26: An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years, growing at an "inflation-compensator" rate of 3%, and sells for \$806,070. What is the interest rate?

- **Q27:** The prevailing interest rate is 10%/year. If you put aside $1,000,000 to cover 18 years of expenses, how much could you draw down each year?

- **Q28:** The prevailing interest rate is 10%/year. If you want to draw $100,000 to cover 18 years of expenses, how much would you have to set aside?

- **Q29:** Assume that our firm has stopped growing in real terms, and the current interest rate is 6% per annum. The inflation rate is 2% per annum. This year, we earned $100,000. What is the value of the firm? (Do it over-the-envelope, and exact [what is the first cash flow?])

- **Q30:** In 2003, MSFT had a P/E earnings ratio of 31. Its cost of capital was roughly 10%. If MSFT lasts forever, what does the market believe its implicit growth will be?

How To Cheat on Loans

An example drawn from an actual automobile loan agreement: The advertisement claimed,

> *12 month car loans. Only 9%!*

Here is how this car dealer calculated the payments on a $10,000 car: A 12-month $10,000 loan, at 9% implies that you owe $10,900. Twelve equal payments come out to $\$10,900/12 \approx \$908.33$ per month.

Agreed?

Q31: What should be the PV of the loan?

Q32: If you took out a loan from the bank at a true interest rate of 9% (8.6% quoted as compounded monthly), how much would the bank have asked you to pay each month?

This is how ordinary loans (such as mortgages) compute interest rates.

Q33: Whence the difference?

The Importance of Time in Annuities

N/A

- **Q34:** Should anyone use perpetuities?

- **Q35:** What fraction of a perpetuity's value comes from the first t years? I.e., how reasonable an approximation is it to use a perpetuity in a meeting for a quick answer?

This tells us how much we are losing if we ignore later years. Put differently, how good an approximation is a perpetuity to an annuity?

Omitted

- Proof of Formulas.
- Formula for A Growing Annuity:

  A growing annuity pays $C \cdot (1+g)^{t-2}$ per year starting in period 1 for T periods. Its present value is given by

  $$\text{PV} = \sum_{t=1}^{T} \left[\frac{(1+g)^{t-2}}{(1+r)^t}\right] \cdot C = \left(\frac{C}{r-g}\right) \cdot \left[1 - \frac{(1+g)^T}{(1+r)^T}\right]$$

  If you do not wish to memorize the growing annuity formula...look it up in the book. It is what I do. Now you have seen it, and you know where to look it up. Finito.

  You have an annuity that pays $100 in period 1. The annuity grows at a rate of 6% per year, and pays off until and including period 10. The discount rate equals 8%.

Q36: What is the present value?

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