CHAPTER 10[§]

Statistics of Portfolios

This chapter appears in the Survey text only.

THE previous chapter explained how to measure risk and reward for an investment—standard deviation and expected rate of return. This chapter explains how to measure these statistics in the context of portfolios. It may be the most tedious chapter in the book. But it is also important: these formulas will be used in subsequent chapters, where you will want to find the risks and rewards of many portfolios. You cannot understand investments without having read this chapter—without understanding the rules for working with statistics in a portfolio context. Your ultimate goal in reading this chapter is to ingest (understand how to use) the methods described in Table 10.5.

Admittedly, some of this chapter "overdoes" it—it tries to explain where the algebra comes from, even though in the end you only need to know the rules. Consider the extra pages to be "useful reference."

Table 10.1: Historical Rates of Returns and Statistics for S&P500, IBM, Sony, and a portfolio P

Historical Annual Rates of Returns

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$
1991	+0.263	−0.212	−0.103	−0.1758	1997	+0.310	+0.381	+0.391	+0.3842
1992	+0.045	−0.434	−0.004	−0.2903	1998	+0.270	+0.762	−0.203	+0.4407
1993	+0.071	+0.121	+0.479	+0.2400	1999	+0.195	+0.170	+2.968	+1.1028
1994	−0.015	+0.301	+0.135	+0.2457	2000	−0.101	−0.212	−0.511	−0.3116
1995	+0.341	+0.243	+0.105	+0.1969	2001	−0.130	+0.423	−0.348	+0.1659
1996	+0.203	+0.658	+0.077	+0.4647	2002	−0.234	−0.357	−0.081	−0.2647
					Mean over all 12 years	+0.101	+0.154	0.242	+0.183

Quoted as Deviations from the Mean

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$
1991	+0.1620	−0.3661	−0.3448	−0.3590	1997	+0.2090	+0.2273	+0.1485	+0.2010
1992	−0.0565	−0.5874	−0.2458	−0.4735	1998	+0.1656	+0.6086	−0.4448	+0.2575
1993	−0.0305	−0.0330	+0.2364	+0.0568	1999	+0.0942	+0.0163	+2.7261	+0.9196
1994	−0.1165	+0.1474	−0.1073	+0.0625	2000	−0.2025	−0.3658	−0.7529	−0.4948
1995	+0.2400	+0.0892	−0.1374	+0.0137	2001	−0.2315	+0.2693	−0.5904	−0.0173
1996	+0.1015	+0.5046	−0.1648	+0.2815	2002	−0.3348	−0.5105	−0.3228	−0.4479
					Mean (of Deviations) over all 12 years	0.0	0.0	0.0	0.0

Statistics: The covariances (and variances) computed in the previous chapter:

$$\mathcal{V}ar(\tilde{r}_{S\&P500}) = 0.0362\ , \quad Cov(\tilde{r}_{S\&P500}, \tilde{r}_{IBM}) = 0.0330\ , \quad Cov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony}) = 0.0477$$

$$Cov(\tilde{r}_{IBM}, \tilde{r}_{S\&P500}) = 0.0330\ , \quad \mathcal{V}ar(\tilde{r}_{IBM}) = 0.1503\ , \quad Cov(\tilde{r}_{IBM}, \tilde{r}_{Sony}) = 0.0218$$

$$Cov(\tilde{r}_{Sony}, \tilde{r}_{S\&P500}) = 0.0477\ , \quad Cov(\tilde{r}_{Sony}, \tilde{r}_{IBM}) = 0.0218\ , \quad \mathcal{V}ar(\tilde{r}_{Sony}) = 0.8149$$

The new portfolio P covariances, computed from the twelve historical returns above, are

$$Cov(\tilde{r}_{S\&P500}, \tilde{r}_{P}) = 0.0379\ , \quad Cov(\tilde{r}_{IBM}, \tilde{r}_{P}) = 0.1075$$

$$Cov(\tilde{r}_{Sony}, \tilde{r}_{P}) = 0.2862\ , \quad Cov(\tilde{r}_{P}, \tilde{r}_{P}) = 0.1671$$

The standard deviations are therefore

$$Sdv(\tilde{r}_{S\&P500}) = 19.0\%\ , \quad Sdv(\tilde{r}_{IBM}) = 38.8\%\ , \quad Sdv(\tilde{r}_{Sony}) = 90.3\%\ , \quad Sdv(\tilde{r}_{P}) = 40.9\%\ .$$

The return of portfolio P is $\tilde{r}_{P} = 66.7\% \cdot \tilde{r}_{IBM} + 33.3\% \cdot \tilde{r}_{Sony}$. Note: to keep investment weights constant at 66.7% and 33.3%, you must rebalance the portfolio every year.

10·1 Two Investment Securities

Table 10.1 summarizes what you did in the previous chapter, plus it adds statistics for a portfolio P.

Our goal in this section is to explore the properties of a portfolio P that invests twice as much into IBM as it invests into Sony. Table 10.1 begins with all the information you computed in the previous chapter: historical rates of return, means, deviations from the means, variances, covariances, and standard deviations. The only novelty is that the performance of a portfolio P in each year is now also in the table—as if it were a stock that you could have purchased. The table also repeats the calculations for the covariances and variances for P. Please check the calculations—do not go on until you have convinced yourself both that you can compute the basic statistics for P yourself, and that the table contains no mistakes.

Solve Now!

Q 10.1 Compute the $\tilde{r}_P$ related statistics from the twelve historical rates of returns in Table 10.1.

Q 10.2 Is there an error in Table 10.1? If so, can you find it?

10·1.A. Expected Rates of Returns

The expected rate of return is the investment-weighted average.

You knew this one even before you ever opened my book. If you expect 20% in your first stock and 30% in your second stock, and you invest half in each, you expect to earn 25%. In our case, in which portfolio P is defined by

$$P \equiv (2/3 \text{ IBM}, 1/3 \text{Sony})$$

the expected rate of return on P of 18.3%. You could directly work this out from the 12 observations in the time-series in the first panel of Table 10.1, so you should confirm now that it is also the investment-weighted average of the expected rates of return on our portfolio,

$$\begin{aligned} \mathcal{E}(\tilde{r}_P) &= \mathcal{E}(w_{IBM} \cdot \tilde{r}_{IBM} + w_{Sony} \cdot \tilde{r}_{Sony}) \\ &= w_{IBM} \cdot \mathcal{E}(\tilde{r}_{IBM}) + w_{Sony} \cdot \mathcal{E}(\tilde{r}_{Sony}) \\ &= 66.7\% \cdot 15.4\% + 33.3\% \cdot 24.2\% = 18.3\% \end{aligned}$$

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_P \equiv w_1 \cdot r_1 + w_2 \cdot r_2$.

You can work with expected rates of return of a portfolio by taking the investment-weighted average of its constituents, as follows:

$$\begin{aligned} \mathcal{E}(\tilde{r}_P) &= \mathcal{E}(w_1 \cdot \tilde{r}_1 + w_2 \cdot \tilde{r}_2) \\ &= w_1 \cdot \mathcal{E}(\tilde{r}_1) + w_2 \cdot \mathcal{E}(\tilde{r}_2) \end{aligned}$$

10·1.B. Covariance

Compute the covariance the slow way: from the twelve historical returns.

A more interesting question is about the covariance of our portfolio P (2/3 in IBM, 1/3 in Sony) with the returns on some other portfolio, say, the S&P500. In Table 10.1, the covariance worked out from the twelve historical returns is

$$Cov(\ \tilde{r}_{S\&P500}\ ,\ \tilde{r}_{P}\ ) = \frac{(-0.3590)\cdot(+0.1620)+\cdots+(-0.4479)\cdot(-0.3348)}{11} = 0.0379$$

(This computation works with the deviations from the means.) But how does our P portfolio's covariance with the S&P500 (0.0379) relate to the covariances of its two portfolio components with the S&P500 (0.0330 for IBM, and 0.0477 for Sony)?

The covariance of the rate of return with something else is the investment-weighted average.

In the previous section, you learned that the expected rate of return on our portfolio is the investment-weighted average of the expected rates of the portfolio constituents. Can you do the same for covariance—i.e., is the portfolio covariance equal to the weighted sum of its portfolio constituents' covariances? Yes!

$$\begin{array}{ccccc} 66.7\%\cdot 0.0330 & + & 33.3\%\cdot 0.0477 & = & 0.0379 \\ w_{IBM}\cdot Cov(\ \tilde{r}_{IBM}\ ,\ \tilde{r}_{S\&P500}\ ) & + & w_{Sony}\cdot Cov(\ \tilde{r}_{Sony}\ ,\ \tilde{r}_{S\&P500}\ ) & = & Cov(\ \tilde{r}_{P}\ ,\ \tilde{r}_{S\&P500}\ ) \end{array} \tag{10.1}$$

If you want to find out the covariance of our portfolio with the S&P500 (or any other security), all you need to do is compute the weighted average of its constituents. You do not need to recompute the covariance from scratch in the tedious multi-step manner if you want to experiment with different portfolio weights!

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_P \equiv w_1 \cdot r_1 + w_2 \cdot r_2$.

You can work with the covariances of your portfolio P with any other portfolio X by taking the investment-weighted average of its constituents, as follows:

$$\begin{aligned} Cov(\ \tilde{r}_P\ ,\ \tilde{r}_X\ ) &= Cov(\ w_1\cdot\tilde{r}_1 + w_2\cdot\tilde{r}_2\ ,\ \tilde{r}_X\ ) \\ &= w_1\cdot Cov(\ \tilde{r}_1\ ,\ \tilde{r}_X\ ) + w_2\cdot Cov(\ \tilde{r}_2\ ,\ \tilde{r}_X\ ) \end{aligned}$$

If you like our σ notation, you can write this as

$$\sigma_{P,X} = w_1\cdot\sigma_{1,X} + w_2\cdot\sigma_{2,X}$$

As usual, we just omit the $\tilde{r}$ in the sigma subscripts, so that we avoid double subscripts. (Incidentally, this covariance law is only interesting, because we need it to work out the laws for variance and beta.)

10·1.C. Beta

Our next question is: what is the beta of our portfolio P with respect to another security—here again the S&P500? That is, how can you express the portfolio beta for P in terms of the betas of its two constituents (2/3 in IBM, 1/3 in Sony). Recall from the previous chapter how beta is defined: you divide the covariance between $\tilde{X}$ and $\tilde{Y}$ by the variance of the $\tilde{X}$ variable,

The covariance of the rate of return with something else is the investment-weighted average.

$$
\begin{aligned}
\beta_{\text{IBM,S\&P500}} &= \frac{Cov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} = \frac{0.0330}{0.0362} \approx 0.91 \\
\beta_{\text{Sony,S\&P500}} &= \frac{Cov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} = \frac{0.0477}{0.0362} \approx 1.32 \\
\beta_{\text{P,S\&P500}} &= \frac{Cov(\ \tilde{r}_{\text{P}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} = \frac{0.0379}{0.0362} \approx 1.05 \\
\beta_{Y,X} &= \frac{Cov(\ \tilde{r}_X\ ,\ \tilde{r}_Y\ )}{\mathcal{V}ar(\tilde{r}_X)}
\end{aligned}
\tag{10.2}
$$

Note that the second subscript on beta is the variance denominator, and that we again omit the $\tilde{r}$ in the beta subscripts so as to avoid double subscripts.

In the previous subsections, you learned that both the expected rate of return and the covariance of our portfolio with another portfolio are the investment-weighted statistics of the portfolio constituents, respectively. Can you do the same for beta—i.e., is the portfolio beta equal to the weighted sum of its portfolio betas? Yes!

The beta of the rate of return with something else is the investment-weighted average.

$$
\begin{aligned}
66.7\% \cdot 0.91 \quad &+ \quad 33.3\% \cdot 1.32 \quad = \quad 1.05 \\
w_{\text{IBM}} \cdot \beta_{\text{IBM,S\&P500}} \quad &+ \quad w_{\text{Sony}} \cdot \beta_{\text{Sony,S\&P500}} \quad = \quad \beta_{\text{P,S\&P500}}
\end{aligned}
$$

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_{\text{P}} \equiv w_1 \cdot r_1 + w_2 \cdot r_2$.

You can work with betas of a portfolio by taking the investment-weighted average of its constituents, as follows:

$$
\begin{aligned}
\beta_{\text{P,X}} &= \beta_{(w_1 \cdot \tilde{r}_1 + w_2 \cdot \tilde{r}_2\ ,\ \tilde{r}_{\text{X}})} \\
&= w_1 \cdot \beta_{1,\text{X}} + w_2 \cdot \beta_{2,\text{X}}
\end{aligned}
$$

SIDE NOTE

Two points. First, because you know the covariance law, you can easily show this algebraically,

$$
\begin{aligned}
\beta_{\tilde{r}_{\text{P}},\text{S\&P500}} &= \frac{Cov(\ \tilde{r}_{\text{P}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= \frac{Cov(\ w_{\text{IBM}} \cdot \tilde{r}_{\text{IBM}} + w_{\text{Sony}} \cdot \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= \frac{w_{\text{IBM}} \cdot Cov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{S\&P500}}\ ) + w_{\text{Sony}} \cdot Cov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= w_{\text{IBM}} \cdot \frac{Cov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} + w_{\text{Sony}} \cdot \frac{Cov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= w_{\text{IBM}} \cdot \beta_{\text{IBM , S\&P500}} + w_{\text{Sony}} \cdot \beta_{\text{Sony , S\&P500}}
\end{aligned}
$$

Second, $\beta_{X,P} \neq w_1 \cdot \beta_{X,1} + w_2 \cdot \beta_{X,2}$. Weighted averaging works only for the first beta subscript, not the second.

10·1.D. Variance

The variance of the rate of return is not the investment-weighted average.

Our next question is: how does the variance of the rate of return of our portfolio relate to the covariances of its constituents? This time, our trick does not work. The variance of a portfolio is *not* the investment-weighted average of the portfolio constituents:

$$0.1671 \neq 66.7\% \cdot 0.1503 + 33.3\% \cdot 0.8149 = 0.3719$$
$$\mathcal{V}ar(\tilde{r}_P) \neq w_{IBM} \cdot \mathcal{V}ar(\tilde{r}_{IBM}) + w_{Sony} \cdot \mathcal{V}ar(\tilde{r}_{Sony})$$

(Incidentally, the variance of the portfolio is much lower than just the average of its constituents; you will explore this in great detail in later chapters—after you learn how to work with variances.)

Eliminate the first $\tilde{r}_P$ from the variance...ahem, the covariance with itself.

To find out how the variance can be decomposed, you now have to cleverly use our covariance law. You already know that the variance of a random variable is the covariance with itself—and you have even computed it from the twelve historical returns,

$$\mathcal{V}ar(\tilde{r}_P) = Cov(\tilde{r}_P, \tilde{r}_P) = 0.1671$$

Now, drop in the definition of our portfolio, but just once,

$$\mathcal{V}ar(\tilde{r}_P) = Cov(66.7\% \cdot \tilde{r}_{IBM} + 33.3\% \cdot \tilde{r}_{Sony}, \tilde{r}_P) = 0.1671$$

and use our covariance law to pull out the weights and to create a sum,

$$\begin{aligned}\mathcal{V}ar(\tilde{r}_P) &= Cov(66.7\% \cdot \tilde{r}_{IBM} + 33.3\% \cdot \tilde{r}_{Sony}, \tilde{r}_P) \\ &= 66.7\% \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_P) + 33.3\% \cdot Cov(\tilde{r}_{Sony}, \tilde{r}_P)\end{aligned}$$

Table 10.1 has these covariances, so you can check that we have not committed a mistake yet,

$$\mathcal{V}ar(\tilde{r}_P) = 66.7\% \cdot 0.1075 + 33.3\% \cdot 0.2862 = 0.1671$$
$$w_{IBM} \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_P) + w_{Sony} \cdot Cov(\tilde{r}_{Sony}, \tilde{r}_P)$$

You are still getting the same number through algebra that you obtained by direct computation of the variance (from the twelve historical rates of return in Table 10.1).

Eliminate the second $\tilde{r}_P$ from each term.

Now you must handle each of these two covariance terms by themselves: use the covariance law to pull out the weights and distribute the sum, and substitute in the covariance inputs from Table 10.1,

$$\begin{aligned}Cov(\tilde{r}_{IBM}, \tilde{r}_P) &= Cov(\tilde{r}_{IBM}, 66.7\% \cdot \tilde{r}_{IBM} + 33.3\% \cdot \tilde{r}_{Sony}) \\ &= 66.7\% \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_{IBM}) + 33.3\% \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_{Sony}) \\ &= 66.7\% \cdot 0.1503 + 33.3\% \cdot 0.0218 = 0.1075 \\ Cov(\tilde{r}_{Sony}, \tilde{r}_P) &= Cov(\tilde{r}_{Sony}, 66.7\% \cdot \tilde{r}_{IBM} + 33.3\% \cdot \tilde{r}_{Sony}) \\ &= 66.7\% \cdot Cov(\tilde{r}_{Sony}, \tilde{r}_{IBM}) + 33.3\% \cdot Cov(\tilde{r}_{Sony}, \tilde{r}_{Sony}) \\ &= 66.7\% \cdot 0.0218 + 33.3\% \cdot 0.8149 = 0.2862\end{aligned}$$

(Again, you can check that the algebra is right, because using the information from Table 10.1 gives you the same answers for $Cov(\tilde{r}_{IBM}, \tilde{r}_P)$ and $Cov(\tilde{r}_{Sony}, \tilde{r}_P)$.) Now put it all together,

$$
\begin{aligned}
Var(\tilde{r}_P) &= 0.1671 \\
&= 66.7\% \cdot 0.1075 + 33.3\% \cdot 0.2862 \\
&= w_{IBM} \cdot [Cov(\tilde{r}_{IBM}, \tilde{r}_{S\&P500})] + w_{Sony} \cdot [Cov(\tilde{r}_{Sony}, \tilde{r}_{S\&P500})] \\
&= 66.7\% \cdot \overbrace{[66.7\% \cdot 0.1503 + 33.3\% \cdot 0.0218]}^{0.1075} \\
&\quad + 33.3\% \cdot \underbrace{[66.7\% \cdot 0.0218 + 33.3\% \cdot 0.8149]}_{0.2862} \\
&= w_{IBM} \cdot [w_{IBM} \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_{S\&P500}) + w_{Sony} \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_{Sony})] \\
&\quad + w_{Sony} \cdot [w_{Sony} \cdot Cov(\tilde{r}_{Sony}, \tilde{r}_{S\&P500}) + w_{Sony} \cdot Cov(\tilde{r}_{Sony}, \tilde{r}_{Sony})]
\end{aligned}
$$

Tedious substitutions—but no higher math required. Take the last expression, multiply out (remembering that the covariance of anything with itself is the variance), realize that the two terms that contain both Sony and IBM are the same, rearrange the terms, and you get

$$
\begin{aligned}
Var(\tilde{r}_P) &= (66.7\%)^2 \cdot 0.1503 + (33.3\%)^2 \cdot 0.8149 + 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218 \\
&= 0.1671, \\
Var(\tilde{r}_P) &= w_{IBM}^2 \cdot Var(\tilde{r}_{IBM}) + w_{Sony}^2 \cdot Var(\tilde{r}_{Sony}) + 2 \cdot w_{Sony} \cdot w_{IBM} \cdot Cov(\tilde{r}_{Sony}, \tilde{r}_{IBM})
\end{aligned}
\tag{10.3}
$$

You are done: this is how the variance of a portfolio is expressed in terms of the variances and covariances of its constituent securities. It is the sum of the variances, each multiplied by its weight squared, plus two times each weight times the pairwise covariance. And you know this is right, because the answer is still the same 0.1671 that you computed directly from the historical rates of return for portfolio P in Table 10.2!

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_P \equiv w_1 \cdot r_1 + w_2 \cdot r_2$.

You can work with variances as follows:

$$
Var(\tilde{r}_P) = w_1^2 \cdot Var(\tilde{r}_1) + w_2^2 \cdot Var(\tilde{r}_2) + 2 \cdot w_1 \cdot w_2 \cdot Cov(\tilde{r}_1, \tilde{r}_2) \tag{10.4}
$$

This is <u>not</u> the investment-weighted average of its constituents.

If you liked our σ notation, you can write this briefer as

$$
\begin{aligned}
\sigma_P^2 = \sigma_{P,P} &= w_X \cdot \sigma_{X,X} + w_Y \cdot \sigma_{Y,Y} + 2 \cdot w_X \cdot w_Y \cdot \sigma_{X,Y} \\
&= w_X \cdot \sigma_X^2 + w_Y \cdot \sigma_Y^2 + 2 \cdot w_X \cdot w_Y \cdot \sigma_{X,Y}
\end{aligned}
$$

Standard Deviation

To compute the standard deviation, you always have to first compute the variance, and then take the square-root. You *cannot* take any shortcuts here—like computing the average standard deviation of the portfolio's constituents.

Solve Now!

Q 10.3 Assume portfolio S consists of 25% IBM and 75% Sony. Compute the covariance of the rate of return of S with the rates of return on IBM, Sony, and S&P500. (If you feel shaky, do it with both a historical time-series in a table, and with the formulas.)

Q 10.4 Continue with portfolio S. Compute the beta of portfolio S with respect to the S&P500, denoted $\beta_{S,S\&P500}$. (If you feel shaky, do it with a historical time-series in a table, then directly from the covariances, and finally with the beta-combination formula.)

Q 10.5 Continue with portfolio S. Compute its variance. (If you feel shaky, do it with both a historical time-series in a table, and the formulas.)

10·2 Three and More Investment Securities

We now work with portfolios of more than two securities, often expressed in summation notation.

Of course, you rarely want a portfolio with just two investments—usually, your portfolio will have more than two investments. Let me remind you of how summation notation works. As in Chapter 8, we later use i as a counter to enumerate each and every possible investment, from N choices (e.g., stocks). w_i are the investment weights that define the portfolio. The return of any portfolio Q defined by these weights in each and every time period is

$$r_Q = \sum_{i=1}^{N} w_i \cdot r_i$$

but because you do not yet know the returns,

$$\tilde{r}_Q = \sum_{i=1}^{N} w_i \cdot \tilde{r}_i$$

10·2.A. Expected Returns, Covariance, Beta

Expected returns, covariances, and betas can be averaged, using the investment-weighted proportions.

For the three statistics from the previous section that we could average—the expected return, the covariance, and beta—the generalization from two securities to any number (N) of securities is easy: you can just take a weighted sum of all the components, not just of the first two components. This is what our next important box expresses.

IMPORTANT: (For a portfolio Q that consists of w_1 investment in security 1, w_2 investment in security 2, all the way up to w_N investment in security N, and which therefore has a rate of return of $r_Q \equiv w_1 \cdot r_1 + w_2 \cdot r_2 + \cdots + w_N \cdot r_N = \sum_{i=1}^{N} w_i r_i$):

You can work with expected rates of return, covariances, and betas by taking the investment-weighted average of its constituents, as follows:

$$\begin{aligned}
\mathcal{E}(\tilde{r}_Q) &= w_1 \cdot \mathcal{E}(\tilde{r}_1) + w_2 \cdot \mathcal{E}(\tilde{r}_2) + \cdots + w_N \cdot \mathcal{E}(\tilde{r}_N) = \sum_{i=1}^{N} w_i \cdot \mathcal{E}(\tilde{r}_i) \\
\sigma_{P,X} &= w_1 \cdot \sigma_{1,X} + w_2 \cdot \sigma_{2,X} + \cdots + w_N \cdot \sigma_{N,X} = \sum_{i=1}^{N} w_i \cdot \sigma_{i,X} \\
\beta_{P,X} &= w_1 \cdot \beta_{1,X} + w_2 \cdot \beta_{2,X} + \cdots + w_N \cdot \beta_{N,X} = \sum_{i=1}^{N} w_i \cdot \beta_{i,X}
\end{aligned} \tag{10.5}$$

Checking the formulas for a particular portfolio, that we shall use later, too.

Let's use these formulas on a new portfolio Q that has 70% invested in the S&P500, 20% invested in IBM, and 10% invested in Sony. You want to know the expected rate of return of our portfolio ($\mathcal{E}(\tilde{r}_{Q})$), its covariance with S&P500 ($Cov(\tilde{r}_{Q}, \tilde{r}_{S\&P500}) \equiv \sigma_{Q,S\&P500}$), and its beta with respect to S&P500 ($\beta_{Q,S\&P500}$). Corresponding to the formulas' numbering schemes, security 1 is S&P500, security 2 is IBM, and security 3 is Sony. Table 10.1 provided all the necessary expected returns and covariances; betas were in Formula 10.2; and your portfolio investment weights are given.

$$
\begin{aligned}
\tilde{r}_{Q} &= 70\% \cdot \tilde{r}_{S\&P500} + 20\% \cdot \tilde{r}_{IBM} + 10\% \cdot \tilde{r}_{Sony} \\
\mathcal{E}(\tilde{r}_{Q}) &= 70\% \cdot 10.1\% + 20\% \cdot 15.4\% + 10\% \cdot 24.2\% = 12.57\% \\
&= w_{S\&P500} \cdot \mathcal{E}(\tilde{r}_{S\&P500}) + w_{IBM} \cdot \mathcal{E}(\tilde{r}_{IBM}) + w_{Sony} \cdot \mathcal{E}(\tilde{r}_{Sony}) \\
&= \sum_{i=1}^{3} w_i \cdot \mathcal{E}(\tilde{r}_i) = \sum_{S\&P500,IBM,Sony} w_i \cdot \mathcal{E}(\tilde{r}_i) \;; \\
\sigma_{Q,S\&P500} &= 70\% \cdot 0.03622 + 20\% \cdot 0.03298 + 10\% \cdot 0.04772 = 0.03672 \\
&= w_{S\&P500} \cdot \sigma_{S\&P500,S\&P500} + w_{IBM} \cdot \sigma_{IBM,S\&P500} + w_{Sony} \cdot \sigma_{Sony,S\&P500} \\
&= \sum_{i=1}^{3} w_i \cdot \sigma_{i,S\&P500} = \sum_{S\&P500,IBM,Sony} w_i \cdot \sigma_{i,S\&P500} \;; \\
\beta_{Q,S\&P500} &= 70\% \cdot 1.000 + 20\% \cdot 0.910 + 10\% \cdot 1.317 = 1.0138 \\
&= w_{S\&P500} \cdot \beta_{S\&P500,S\&P500} + w_{IBM} \cdot \beta_{IBM,S\&P500} + w_{Sony} \cdot \beta_{Sony,S\&P500} \\
&= \sum_{i=1}^{3} w_i \cdot \beta_{i,S\&P500} = \sum_{S\&P500,IBM,Sony} w_i \cdot \beta_{i,S\&P500}
\end{aligned}
\qquad (10.7)
$$

Table 10.2 gives the historical rates of return of this portfolio next to its constituents. Please confirm from the twelve historical returns for Q and S&P500 that the above three statistics are correct.

DIG DEEPER

Expectations, covariances, and betas are so-called linear function, because

$$f(a+b) = f(a) + f(b) \qquad (10.8)$$

For our expectations, covariances, and beta, a would be defined as $w_1 \cdot \tilde{r}_1$, and b as $w_2 \cdot \tilde{r}_2$. For example, $\mathcal{E}(w_1 \cdot \tilde{r}_1 + w_2 \cdot \tilde{r}_2) = \mathcal{E}(w_1 \cdot \tilde{r}_1) + \mathcal{E}(w_2 \cdot \tilde{r}_2)$.

It is not difficult to prove that if a function is linear, then it works for more than two securities in a portfolio. Just replace b with $c+d$

$$f(a+(c+d)) = f(a) + f(c+d)$$

and apply formula 10.8 again on the $c+d$ term to get

$$f(a+(c+d)) = f(a) + f((c+d)) = f(a) + f(c) + f(d)$$

Solve Now!

Q 10.6 Confirm that the computations for the expected rate of return and the covariance in Formula 10.6 are correct by directly computing these statistics from the historical timeseries in Table 10.2.

Q 10.7 Consider a portfolio T that consists of 25% S&P500, 35% IBM, and 40% Sony. What is its expected rate of return, its covariance with the S&P500, and its beta with respect to the S&P500? (If you feel shaky, compute this from both the twelve historical rates of return on T, and from the formulas.)

Table 10.2: Historical Annual Rates of Returns for portfolio Q

Historical Annual Rates of Returns

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_Q$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_Q$
1991	+0.263	−0.212	−0.103	+0.131	1997	+0.310	+0.381	+0.391	+0.332
1992	+0.045	−0.434	−0.004	−0.056	1998	+0.270	+0.762	−0.203	+0.319
1993	+0.071	+0.121	+0.479	+0.121	1999	+0.195	+0.170	+2.968	+0.468
1994	−0.015	+0.301	+0.135	+0.063	2000	−0.101	−0.212	−0.511	−0.165
1995	+0.341	+0.243	+0.105	+0.298	2001	−0.130	+0.423	−0.348	−0.042
1996	+0.203	+0.658	+0.077	+0.281	2002	−0.234	−0.357	−0.081	−0.243
					Mean over all 12 years	+0.101	+0.154	0.242	+0.1257

Quoted as Deviations from the Mean

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_Q$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_Q$
1991	+0.1620	−0.3661	−0.3448	−0.006	1997	+0.2090	+0.2273	+0.1485	+0.207
1992	−0.0565	−0.5874	−0.2458	−0.182	1998	+0.1656	+0.6086	−0.4448	+0.193
1993	−0.0305	−0.0330	+0.2364	+0.004	1999	+0.0942	+0.0163	+2.7261	+0.342
1994	−0.1165	+0.1474	−0.1073	+0.063	2000	−0.2025	−0.3658	−0.7529	−0.290
1995	+0.2400	+0.0892	−0.1374	+0.172	2001	−0.2315	+0.2693	−0.5904	−0.167
1996	+0.1015	+0.5046	−0.1648	+0.156	2002	−0.3348	−0.5105	−0.3228	−0.369
					Mean (of Deviations) over all 12 years	0.0	0.0	0.0	0.0

The return of portfolio Q is $\tilde{r}_Q = 70\%\cdot\tilde{r}_{S\&P500} + 20\%\cdot\tilde{r}_{IBM} + 10\%\cdot\tilde{r}_{Sony}$.

10·2.B. Variance

Panic Warning!

Here is where it gets more complicated. What is the variance of portfolio Q, consisting of 70% in S&P500, 20% in IBM, and 10% in Sony? Actually, it will get more tedious, but not more complex. All you need to do is to apply the covariance law twice.

I made it easy on us: portfolio Q is a combination of S&P500 and P.

Recall the portfolio Q that invested 66.7% in IBM and 33.3% in Sony. If you invest 70% in S&P500 and 30% in P, you end up with portfolio Q, because the remaining 30% in P are appropriately split ($w_{IBM} = 30\% \cdot 66.7\% = 20\%$ and $w_{Sony} = 30\% \cdot 33.3\% = 10\%$):

$$\begin{aligned}
\tilde{r}_Q &= 70\% \cdot \tilde{r}_{S\&P500} + \quad (\, 20\% \cdot \tilde{r}_{IBM} + 10\% \cdot \tilde{r}_{Sony}\,) \\
&= 70\% \cdot \tilde{r}_{S\&P500} + 30\% \cdot (\, 66.7\% \cdot \tilde{r}_{IBM} + 33.3\% \cdot \tilde{r}_{Sony}\,) \\
&= 70\% \cdot \tilde{r}_{S\&P500} + \quad 30\% \cdot \tilde{r}_P
\end{aligned}$$

You already know how to work with two securities—and you know the other inputs, too.

With only two securities (S&P500 and P) now, you can use our variance formula 10.4:

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_Q) &= \mathcal{V}ar(w_{S\&P500} \cdot \tilde{r}_{S\&P500} + w_P \cdot \tilde{r}_P) \\
&= (w_{S\&P500})^2 \cdot \mathcal{V}ar(\tilde{r}_{S\&P500}) + (w_P)^2 \cdot \mathcal{V}ar(\tilde{r}_P) + 2 \cdot w_{S\&P500} \cdot w_P \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_P) \qquad (10.9) \\
&= (70\%)^2 \cdot \mathcal{V}ar(\tilde{r}_{S\&P500}) + (30\%)^2 \cdot \mathcal{V}ar(\tilde{r}_P) + 2 \cdot 30\% \cdot 70\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_P)
\end{aligned}$$

Actually, you already know all three remaining unknowns: 0.0362 was the variance of the S&P500, given in Table 10.1. You had worked out $\mathcal{V}ar(\tilde{r}_P)$ in Formula 10.3,

$$\mathcal{V}ar(\tilde{r}_P) = (66.7\%)^2 \cdot 0.1503 + (33.3\%)^2 \cdot 0.8149 + 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218 = 0.1671$$

$$\mathcal{V}ar(\tilde{r}_P) = w_{IBM}^2 \cdot \mathcal{V}ar(\tilde{r}_{IBM}) + w_{Sony}^2 \cdot \mathcal{V}ar(\tilde{r}_{Sony}) + 2 \cdot w_{Sony} \cdot w_{IBM} \cdot \mathcal{C}ov(\tilde{r}_{Sony}, \tilde{r}_{IBM})$$

and $Cov(\tilde{r}_P, \tilde{r}_{S\&P500})$ in Formula 10.1,

$$Cov(\tilde{r}_P, \tilde{r}_{S\&P500}) = \underbrace{66.7\% \cdot 0.0330 + 33.3\% \cdot 0.0477}_{w_{IBM} \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_{S\&P500}) + w_{Sony} \cdot Cov(\tilde{r}_{IBM}, \tilde{r}_{S\&P500})} = 0.0379$$

Now just substitute these terms into Formula 10.9 to get

$$\begin{aligned} Var(\tilde{r}_Q) &= (70\%)^2 \cdot Var(\tilde{r}_{IBM}) + (30\%)^2 \cdot Var(\tilde{r}_P) + 2 \cdot 70\% \cdot 30\% \cdot Cov(\tilde{r}_{S\&P500}, \tilde{r}_P) \\ &= (70\%)^2 \cdot 0.0362 \\ &\quad + (30\%)^2 \cdot \underbrace{[(66.7\%)^2 \cdot 0.1503 + (33.3\%)^2 \cdot 0.8149 + 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218]}_{Var(\tilde{r}_P)} \\ &\quad + 2 \cdot 70\% \cdot 30\% \cdot \underbrace{[66.7\% \cdot 0.0330 + 33.3\% \cdot 0.0477]}_{Cov(\tilde{r}_{S\&P500}, \tilde{r}_P)} \\ &= (70\%)^2 \cdot 0.0362 + (30\%)^2 \cdot 0.1671 + 2 \cdot 70\% \cdot 30\% \cdot 0.0379 \\ &= 0.0487 \ . \end{aligned} \tag{10.10}$$

Please confirm this from the twelve annual rates of return for portfolio Q in Table 10.2.

Although we are done—we have our answer for the variance of our portfolio Q—let's do some more algebra "just for fun." Take the middle form from the previous formula,

A lengthy detailed step-by-step rewrite of the answer shows us what the answer really consists of: three variance terms and three pairwise covariance terms.

$$\begin{aligned} Var(\tilde{r}_Q) &= (70\%)^2 \cdot 0.0362 \\ &\quad + (30\%)^2 \cdot [(66.7\%)^2 \cdot 0.1503 + (33.3\%)^2 \cdot 0.8149 + 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218] \\ &\quad + 2 \cdot 70\% \cdot 30\% \cdot [66.7\% \cdot 0.0330 + 33.3\% \cdot 0.0477] \\ &= 0.04869 \ . \end{aligned}$$

Now, multiply the (30%) and $(30\%)^2$ terms into the parentheses, and pull the 30% and 30%% into the adjacent weights,

$$\begin{aligned} &= (70\%)^2 \cdot 0.0362 \\ &\quad + [(30\% \cdot 66.7\%)^2 \cdot 0.1503 + (30\% \cdot 33.3\%)^2 \cdot 0.8149 + 2 \cdot (30\% \cdot 66.7\%) \cdot (30\% \cdot 33.3\%) \cdot 0.0218] \\ &\quad + 2 \cdot 70\% \cdot [(30\% \cdot 66.7\%) \cdot 0.0330 + (30\% \cdot 33.3\%) \cdot 0.0477] \\ &= 0.04869 \ . \end{aligned}$$

Execute the 30% multiplication, multiply in the $2 \cdot 70\%$, eliminate some parentheses, and reorder terms,

$$\begin{aligned} &= (70\%)^2 \cdot 0.0362 + (20\%)^2 \cdot 0.1503 + (10\%)^2 \cdot 0.8149 \\ &\quad + 2 \cdot (20\%) \cdot (10\%) \cdot 0.0218 + 2 \cdot (70\%) \cdot (20\%) \cdot 0.0330 + 2 \cdot (70\%) \cdot (10\%) \cdot 0.0477 \\ &= 0.04869 \ . \end{aligned}$$

Now stare at this formula, and recall the form of the variance formula in 10.4. What does this formula consist of? Well, it is just

↦ *Variance Formula 10.4 on Page 217*

$$\begin{aligned} Var(\tilde{r}_Q) &= w_{S\&P500}^2 \cdot Var(\tilde{r}_{S\&P500}) + w_{IBM}^2 \cdot Var(\tilde{r}_{IBM}) + w_{Sony}^2 \cdot Var(\tilde{r}_{Sony}) \\ &\quad + 2 \cdot w_{IBM} \cdot w_{Sony} \cdot Cov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony}) \\ &\quad + 2 \cdot w_{S\&P500} \cdot w_{IBM} \cdot Cov(\tilde{r}_{S\&P500}, \tilde{r}_{IBM}) \\ &\quad + 2 \cdot w_{S\&P500} \cdot w_{Sony} \cdot Cov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony}) \end{aligned}$$

This generalizes, too!

IMPORTANT: To obtain the variance of a portfolio that invests $w_1, w_2, \ldots, w_N$ into N securities, do the following:

1. For each security, square its weights and multiply it by the variance.
2. For each pair of different securities, multiply two times the first weight times the second weight times the securities' covariance.
3. Add up all these terms.

For example, for three securities, the formula is

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_Q) = {} & w_1^2 \cdot \mathcal{V}ar(\tilde{r}_1) + w_2^2 \cdot \mathcal{V}ar(\tilde{r}_2) + w_3^2 \cdot \mathcal{V}ar(\tilde{r}_3) \\ & + 2 \cdot w_1 \cdot w_2 \cdot Cov(\tilde{r}_1, \tilde{r}_2) \\ & + 2 \cdot w_1 \cdot w_3 \cdot Cov(\tilde{r}_1, \tilde{r}_3) \\ & + 2 \cdot w_2 \cdot w_3 \cdot Cov(\tilde{r}_2, \tilde{r}_3) \end{aligned}$$

For portfolios with many securities, there are relatively more covariance than variance terms.

If you do not believe me, feel free to repeat this exercise for four securities. (The principle remains the same, but it becomes a lot more messy.) Incidentally, the number of covariance terms increases more rapidly than the number of variance terms. With four securities, you will have four variance terms, and six pairwise covariance terms. With ten securities, you will have ten variance terms, and forty-five pairwise covariance terms. For one-hundred stocks, there are one-hundred variance terms and 2,475 covariance terms. With more and more securities in the portfolio, there are fewer and fewer "own return variance" terms, and more and more "return covariance terms." Thus, on first glance, it seems that the overall portfolio variance could be driven more by the many covariance terms than the few variance terms. This will play a major role in the next chapters.

Solve Now!

Q 10.8 Continue with our portfolio T that consists of 25% S&P500, 35% IBM, and 40% Sony. What is its standard deviation of return? Compute this both from the twelve historical rates of return on T, and from the formulas.

10·2.C. Advanced Nerd Section: Variance with N Securities and Double Summations

The Formal Definitions.

Formula 10.5 used $\sum$ notation to write sums more compactly. The point of this section is to show how to write the variance formula with two summations signs—a lot more compactly, and perhaps easier to remember. We shall first write it down—don't panic—and then explain and use it. The variance of the rate of return of a portfolio is

$$\mathcal{V}ar(\tilde{r}_P) = \mathcal{V}ar\left(\sum_{i=1}^{N} w_i \cdot \tilde{r}_i\right) = \sum_{j=1}^{N} \left\{ \sum_{k=1}^{N} [w_j \cdot w_k \cdot Cov(\tilde{r}_j, \tilde{r}_k)] \right\} \tag{10.11}$$

What does this formula mean? Write out the terms in this formula to eliminate the summation signs. Concentrate on one step at a time. Start with the innermost parentheses: j is still unknown, so leave j untouched. Just write out the sum for k, which consists of N terms,

$$\sum_{k=1}^{N}[w_j \cdot w_k \cdot Cov(\tilde{r}_j, \tilde{r}_k)] = \left\{ \underbrace{w_j \cdot w_1 \cdot Cov(\tilde{r}_j, \tilde{r}_1)}_{\text{our } k=1 \text{ term}} + \underbrace{w_j \cdot w_2 \cdot Cov(\tilde{r}_j, \tilde{r}_2)}_{\text{our } k=2 \text{ term}} + \cdots + \underbrace{w_j \cdot w_N \cdot Cov(\tilde{r}_j, \tilde{r}_N)}_{\text{our } k=N \text{ term}} \right\}$$

and plug it back into Formula 10.11,

$$\mathcal{V}ar(\tilde{r}_P) = \sum_{j=1}^{N}\left\{\sum_{k=1}^{N}\left[w_j \cdot w_k \cdot Cov(\tilde{r}_j, \tilde{r}_k)\right]\right\}$$
$$= \sum_{j=1}^{N}\left\{w_j \cdot w_1 \cdot Cov(\tilde{r}_j, \tilde{r}_1) + w_j \cdot w_2 \cdot Cov(\tilde{r}_j, \tilde{r}_2) + \cdots + w_j \cdot w_N \cdot Cov(\tilde{r}_j, \tilde{r}_N)\right\}$$

So far, so good. Now this formula tells you that you have N summation terms, each of which is itself N summation terms, so you have a total of N^2 summation terms in the variance. Do what you just did again—write out the sum for j,

$$\mathcal{V}ar(\tilde{r}_P) = \sum_{j=1}^{N}\left\{w_j \cdot w_1 \cdot Cov(\tilde{r}_j, \tilde{r}_1) + w_j \cdot w_2 \cdot Cov(\tilde{r}_j, \tilde{r}_2) + \cdots + w_j \cdot w_N \cdot Cov(\tilde{r}_j, \tilde{r}_N)\right\} =$$

$$\begin{array}{lclclcl}
\text{our j = 1 term} \rightarrow & & \underline{w_1 \cdot w_1 \cdot Cov(\tilde{r}_1, \tilde{r}_1)} & + & w_1 \cdot w_2 \cdot Cov(\tilde{r}_1, \tilde{r}_2) & + \cdots + & w_1 \cdot w_N \cdot Cov(\tilde{r}_1, \tilde{r}_N) \\
\text{our j = 2 term} \rightarrow & + & w_2 \cdot w_1 \cdot Cov(\tilde{r}_2, \tilde{r}_1) & + & \underline{w_2 \cdot w_2 \cdot Cov(\tilde{r}_2, \tilde{r}_2)} & + \cdots + & w_2 \cdot w_N \cdot Cov(\tilde{r}_2, \tilde{r}_N) \\
\text{our j = 3 term} \rightarrow & + & w_3 \cdot w_1 \cdot Cov(\tilde{r}_3, \tilde{r}_1) & + & w_3 \cdot w_2 \cdot Cov(\tilde{r}_3, \tilde{r}_2) & + \cdots + & w_3 \cdot w_N \cdot Cov(\tilde{r}_3, \tilde{r}_N) \\
 & + & \cdots & + & \cdots & + \cdots + & \cdots \\
\text{our j = } N \text{ term} \rightarrow & + & w_N \cdot w_1 \cdot Cov(\tilde{r}_N, \tilde{r}_1) & + & w_N \cdot w_2 \cdot Cov(\tilde{r}_N, \tilde{r}_2) & + \cdots + & \underline{w_N \cdot w_N \cdot Cov(\tilde{r}_N, \tilde{r}_N)}
\end{array} \tag{10.12}$$

This has everything written out, and no longer needs scary summation notation—here it is the length which makes the formula appear intimidating. Would you rather memorize the long form in Formula 10.12 or the short double summation sign Formula 10.11? They both mean exactly the same thing—the double summation is merely abbreviated notation. I find it easier to remember the short form—if need be, I can always expand it into the long form. Before you forget about the long form, though, is this formula really the same as the step-by-step procedure on Page 222? Look at the terms on the diagonal in Formula 10.12, which are underlined. These are covariances of variables with themselves—which are just the variances multiplied by weights squared. Now look at the off-diagonal terms—each term appears twice, because both multiplication and covariances don't care about order. The big-mess Formula 10.12 can also be expressed as

The double summation formula is just an easy-to-remember abbreviation, and is precise. It says exactly what you already knew: variance and pairwise covariance terms.

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= [\text{sum up N diagonal variance terms}] \\
&\quad + \left[\text{sum up remaining } N^2 - N \text{ covariance terms}\right] \\
&= \left[\text{for each security i, sum up each } w_i^2 \text{ times the i-th variance}\right] \\
&\quad + \left[\text{for each possible pair i and j, sum up twice } w_i \text{ times } w_j \text{ times the i vs. j covariance}\right]
\end{aligned} \tag{10.13}$$

which is exactly what was stated on Page 222: to compute an overall portfolio variance, sum up all the constituent variances ($\mathcal{V}ar(\tilde{r}_i)$), each multiplied by its squared weight (w_i^2); and then add each pairwise covariance ($Cov(\tilde{r}_i, \tilde{r}_j)$), multiplied by two times its two weights ($2 \cdot w_i \cdot w_j$).

IMPORTANT: The mean and variance formulas for portfolios deserve memorizing if you want to concentrate in investments:

$$\mathcal{E}(\tilde{r}_P) = \sum_{i=1}^{N} w_i \cdot \mathcal{E}(\tilde{r}_i)$$
$$\mathcal{V}ar(\tilde{r}_P) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \cdot w_j \cdot Cov(\tilde{r}_i , \tilde{r}_j)$$

If you prefer sigma notation, this is even shorter: $\sigma_{P,P} = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \cdot w_j \cdot \sigma_{i,j}$.

Application of the formula makes it clear that it is the same thing.

Before you forget about double summations, let us just confirm that the formula gives us the same variance for our portfolio Q:

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_Q) &= \sum_{i \in \{S\&P500, IBM, Sony\}} \left\{ \sum_{j \in \{S\&P500, IBM, Sony\}} [w_i \cdot w_j \cdot Cov(\tilde{r}_i , \tilde{r}_j)] \right\} \\
&= \Big[w_{S\&P500} \cdot w_{S\&P500} \cdot Cov(\tilde{r}_{S\&P500} , \tilde{r}_{S\&P500}) + w_{IBM} \cdot w_{S\&P500} \cdot Cov(\tilde{r}_{IBM} , \tilde{r}_{S\&P500}) \\
&\qquad + w_{Sony} \cdot w_{S\&P500} \cdot Cov(\tilde{r}_{Sony} , \tilde{r}_{S\&P500}) \Big] \\
&+ \Big[w_{S\&P500} \cdot w_{IBM} \cdot Cov(\tilde{r}_{S\&P500} , \tilde{r}_{IBM}) + w_{IBM} \cdot w_{IBM} \cdot Cov(\tilde{r}_{IBM} , \tilde{r}_{IBM}) \\
&\qquad + w_{Sony} \cdot w_{IBM} \cdot Cov(\tilde{r}_{Sony} , \tilde{r}_{IBM}) \Big] \\
&+ \Big[w_{S\&P500} \cdot w_{Sony} \cdot Cov(\tilde{r}_{S\&P500} , \tilde{r}_{Sony}) + w_{IBM} \cdot w_{Sony} \cdot Cov(\tilde{r}_{IBM} , \tilde{r}_{Sony}) \\
&\qquad + w_{Sony} \cdot w_{Sony} \cdot Cov(\tilde{r}_{Sony} , \tilde{r}_{Sony}) \Big] \\
&= \big[70\% \cdot 70\% \cdot 0.0362 + 20\% \cdot 70\% \cdot 0.0330 + 10\% \cdot 70\% \cdot 0.0218 \big] \\
&+ \big[70\% \cdot 20\% \cdot 0.0330 + 20\% \cdot 20\% \cdot 0.1503 + 10\% \cdot 20\% \cdot 0.0218 \big] \\
&+ \big[70\% \cdot 10\% \cdot 0.0477 + 20\% \cdot 10\% \cdot 0.0218 + 10\% \cdot 10\% \cdot 0.8149 \big] \\
&= 0.04869
\end{aligned} \tag{10.14}$$

and you have the answer you already knew!

10·2.D. Another Variance Example: PepsiCo, CocaCola, and Cadbury

The table provides only statistics for the rates of return, not the series themselves.

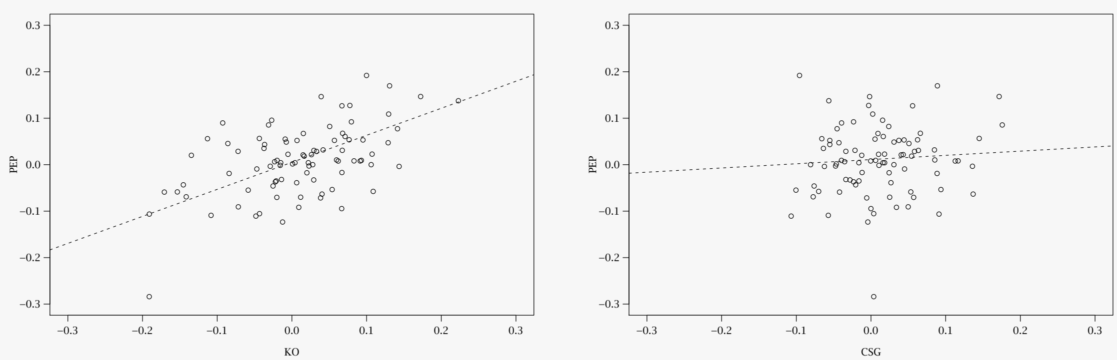
Let us now use portfolio formulas on a second example with monthly data, based on the historical means, standard deviations, and correlations of Coca Cola, PepsiCo, and Cadbury Schweppes, over the 1995-2002 period. This example is deliberately reminiscent of the example from the previous section, but we are now given only the correlations, not the detailed historical return series themselves. Table 10.3 shows that the average return of each of these three stocks was about 1% per month, or 10%–12% per year. The monthly means were significantly lower than the monthly standard deviations. Over the sample period, Cadbury Schweppes had higher performance and lower risk than either PepsiCo or Coca Cola. Note also how high the 53%

Table 10.3: PepsiCo, Coca Cola, and Cadbury Schweppes Monthly Return Statistics, from September 1995 to August 2002.

	Investment	Means	Standard Deviations	Correlations PEP	KO	CSG
1	PEP	0.83%	7.47%	100.0%		
2	KO	0.90%	8.35%	53.2%	100.0%	
3	CSG	1.19%	6.29%	10.8%	9.9%	100.0%

correlation between PepsiCo and Coca Cola is, especially relative to the 10.8% and 9.9% Cadbury Schweppes correlations. Figure 10.1 plots the data points. Coca Cola stock seems to behave more like PepsiCo stock than like Cadbury Schweppes stock.

Figure 10.1: 1,765 Daily Stock Returns of PepsiCo vs. Coca Cola, and PepsiCo vs. Cadbury Schweppes from August 1995 to August 2002



As always, assume that historical means, standard deviations, and correlations are indicative of future means, standard deviations, and correlations. Now determine the risk and reward of a portfolio B defined by

A real-world example that computes the risk and reward of a portfolio of three stocks.

$$B \equiv (20\% \text{ in PEP}, 30\% \text{ in KO, and } 50\% \text{ in CSG})$$

The unknown rate of return on this portfolio is

$$\tilde{r}_B = 20\% \cdot \tilde{r}_{PEP} + 30\% \cdot \tilde{r}_{KO} + 50\% \cdot \tilde{r}_{CSG}$$

The reward is easy. It is

$$\mathcal{E}(\tilde{r}_B) = 20\% \cdot 0.83\% + 30\% \cdot 0.90\% + 50\% \cdot 1.19\% = 1.03\%$$

$$\mathcal{E}(\tilde{r}_B) = 20\% \cdot \mathcal{E}(\tilde{r}_{PEP}) + 30\% \cdot \mathcal{E}(\tilde{r}_{KO}) + 50\% \cdot \mathcal{E}(\tilde{r}_{CSG})$$

The portfolio risk is more difficult. As inputs, you need covariances, not standard deviations or correlations. The covariances of variables with themselves (i.e., the variances) of the three stocks can be computed from the standard deviations, by squaring:

$$\sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{PEP}}} = (0.0747)^2 = 0.005585$$

$$\sigma_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{KO}}} = (0.0835)^2 = 0.006967$$

$$\sigma_{\tilde{r}_{\text{CSG}},\tilde{r}_{\text{CSG}}} = (0.0629)^2 = 0.003956$$

The covariances have to be computed from the correlations. Recall Formula 9.1,

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1 \cdot \sigma_2} \iff \sigma_{1,2} = \rho_{1,2} \cdot \sigma_1 \cdot \sigma_2$$

With this formula, you can compute the covariances,

$$\sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{KO}}} = \rho_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{KO}}} \cdot \sigma_{\tilde{r}_{\text{PEP}}} \cdot \sigma_{\tilde{r}_{\text{KO}}} = 0.532 \cdot 0.0747 \cdot 0.0835 = 0.003318$$

$$\sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{CSG}}} = \rho_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{CSG}}} \cdot \sigma_{\tilde{r}_{\text{PEP}}} \cdot \sigma_{\tilde{r}_{\text{CSG}}} = 0.108 \cdot 0.0747 \cdot 0.0629 = 0.000507$$

$$\sigma_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{CSG}}} = \rho_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{CSG}}} \cdot \sigma_{\tilde{r}_{\text{KO}}} \cdot \sigma_{\tilde{r}_{\text{CSG}}} = 0.099 \cdot 0.0835 \cdot 0.0629 = 0.000522$$

You now have all inputs that you need to compute the portfolio return variance:

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_{\text{B}}) &= \mathcal{V}ar\left(20\% \cdot \tilde{r}_{\text{PEP}} + 30\% \cdot \tilde{r}_{\text{KO}} + 50\% \cdot \tilde{r}_{\text{CSG}}\right) \\
&= (20\%)^2 \cdot \sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{PEP}}} + (30\%)^2 \cdot \sigma_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{KO}}} + (50\%)^2 \cdot \sigma_{\tilde{r}_{\text{CSG}},\tilde{r}_{\text{CSG}}} \\
&\quad + 2 \cdot 20\% \cdot 30\% \cdot \sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{KO}}} + 2 \cdot 30\% \cdot 50\% \cdot \sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{CSG}}} + 2 \cdot 20\% \cdot 50\% \cdot \sigma_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{CSG}}} \\
&= (20\%)^2 \cdot 0.005585 + (30\%)^2 \cdot 0.006967 + (50\%)^2 \cdot 0.003956 \\
&\quad + 2 \cdot 20\% \cdot 30\% \cdot 0.003318 + 2 \cdot 30\% \cdot 50\% \cdot 0.000507 + 2 \cdot 20\% \cdot 50\% \cdot 0.000522 \\
&= 0.002496
\end{aligned}$$

Therefore, the risk of portfolio B is

$$\mathcal{S}dv(\tilde{r}_{\text{B}}) = \sqrt{\mathcal{V}ar(\tilde{r}_{\text{B}})} = \sqrt{0.002496} = 5.00\%$$

Note that this standard deviation is lower than the standard deviation of each of the three stocks by themselves (Table 10.3). This is caused by "diversification," which is explored in great detail in the next chapter.

Solve Now!

Q 10.9 Compute the standard deviation (risk) of portfolio B from Table 10.2 to 4 digits. Is it the same as what we computed in the text?

Q 10.10 Compute the expected rate of return (reward) and standard deviation (risk) for another portfolio, called BF, that invests 10.7% in the S&P500, 64.5% in IBM and 24.8% in Sony. Compute this both from the formulas and from a historical rate of return series of this portfolio.

Q 10.11 Compute the risk and reward for portfolio CF:

$$w_{\text{S\&P500}} = 0.5, \quad w_{\text{IBM}} = 0.5, \quad w_{\text{Sony}} = 0$$

Q 10.12 Compute the risk and reward for portfolio DF:

$$w_{\text{S\&P500}} = 0.614, \quad w_{\text{IBM}} = 0.288, \quad w_{\text{Sony}} = 0.98$$

Q 10.13 What is the risk and reward of a portfolio EF that invests 10% in PEP, 10% in KO, and 80% in CSG?

Q 10.14 What is the slope (beta) of the lines in Figure 10.1?

10·3 Historical Statistics For Some Asset-Class Index Portfolios

This section describes the 1997-2002 investment performance of some Vanguard funds.

Enough with statistical torture! Let's look at the historical performances of some realistic investment portfolios. Table 10.4 describes historical rates of return from portfolios managed by **Vanguard** over the January 1997 to October 2002 period. (**Vanguard** is a prominent low-cost provider of **index funds**.) Each Vanguard fund purchases a large number of securities, often simply everything that qualifies in an asset category (e.g., all suitable bonds, all European stocks, all real estate investment trusts, etc.), and without much attempt at picking winners within each class. Naturally, although you really are interested in forward-looking statistics, standing here today, all you have are historical statistics. So, look at the properties of historical rates of return of these portfolios.

SIDE NOTE

Experience shows that historical means are not good predictors of future means, but historical standard deviations and betas are good predictors of their future equivalents. See the local Nerd Appendix a.

Bonds offered only low rates of return.

The first statistic that this chapter described was the mean. The **historical mean** (also called **sample mean**) of the monthly rates of return describes how you would have fared on average. The second column in Table 10.4 shows that over the sample period, the short-term government bond fund earned a rate of return of about 60 basis points (*per year*). The intermediate government bond fund earned 100 basis points; municipal bonds earned 80 basis points; corporate junk bonds earned 180 basis points; corporate convertible bonds earned about 2.4%; and so on.

Equities offered higher rates of return.

Continuing on to pure equity (stock) investments, you can see that the 500 large stocks in Vanguard's S&P500 fund earned about 4.9% per year. The tax-managed version of the same investment strategy minimizes trading (to minimize capital gains). It did even better than the unmanaged version, earning a rate of return of 6.0% per year. **Value firms** are large and unexciting companies, and **growth firms** are small, fast-growing, exciting companies. Yet, following a long-standing historical trend, value firms earned higher rates of return than growth firms. (During this particular sample period, small growth firms did best, though.) Neither could outperform the S&P500 over the sample period. Among industries, **health care firms** earned the highest rates of return, and **utilities** firms earned the lowest rates of return. International investors fared especially poorly in this sample period: Japanese stocks in particular lost 7.4% *per annum* over the sample period.

SIDE NOTE

The Vanguard S&P500 index portfolio outperformed the S&P500 index itself (second to last line in the table). This is presumably [a] because the fund also received dividends, which are not counted in the S&P index itself; and [b] because the fund could lend out securities to short-sellers and thereby earn some extra return. Incidentally, to reduce transaction costs, index funds usually do not hold the 500 stocks in the exact right proportion, which causes further **tracking error**—a deviation of the performance of the index fund's rate of return from the index percent change.

Equities were riskier than bonds.

The second set of statistics that this chapter described included the standard deviation. The third column in Table 10.4 shows that short-term and medium-term government bonds and municipal bonds were exceptionally safe. Their monthly rates of return varied only a little over the sample period. A large cluster of investment strategies had risks of about 15% to 25% per year, including the overall S&P500 stock market index investment strategy. The riskiest stock market investment strategy in the sample period would have been U.S. Gold and Metals, whereas the safest would have been **Real Estate Investment Trust** (**REITs**) and **Utilities** firms.

Table 10.4: Asset Class Portfolios, Based on Monthly Rates of Returns, Annualized, January 1997 to October 2002.

	Annualized Returns				Monthly Returns					Ann.Market-Model	
Asset	Mean	StdDev	%Neg	%Pos	Worst	Q2	Median	Q3	Best	α_i	$\beta_{i,S\&P}$
govbonds: short-term	0.6%	2.0%	43%	51%	−1.5%	−0.3%	0.1%	0.4%	1.2%	0.7%	−0.0
govbonds: intermediate	1.0%	4.4%	44%	53%	−3.2%	−0.7%	0.2%	0.8%	2.9%	1.2%	-0.0
bond: long-term munis	0.8%	4.5%	43%	54%	−3.1%	−1.1%	0.4%	0.9%	2.7%	1.0%	-0.0
bond: high-yield (junk) bonds	1.8%	16.2%	46%	54%	−15.0%	−2.3%	0.4%	2.9%	9.9%	−1.5%	0.7
bond: corporate convertibles	2.4%	15.4%	50%	49%	−12.8%	−2.7%	−0.2%	3.2%	10.6%	−0.5%	0.6
u.s. s&p500	4.9%	18.5%	46%	54%	−14.5%	−3.0%	0.7%	4.7%	9.5%	0.1%	1.0
u.s. tax managed	6.0%	21.3%	46%	54%	−17.6%	−3.4%	0.4%	5.2%	9.6%	0.6%	1.1
u.s. value firms	1.6%	18.1%	46%	53%	−16.1%	−3.2%	0.3%	3.8%	8.6%	−2.7%	0.9
u.s. growth firm	−2.1%	25.7%	49%	51%	−21.7%	−5.2%	0.8%	5.6%	15.2%	−8.2%	1.3
u.s. small growth firms	7.7%	25.5%	46%	54%	−19.3%	−4.6%	0.3%	6.3%	20.1%	3.1%	1.0
u.s. small-cap firms	2.7%	22.6%	49%	51%	−19.3%	−4.3%	0.3%	5.2%	16.5%	−1.3%	0.8
u.s. energy firms	5.2%	23.7%	53%	47%	−18.0%	−4.1%	−0.5%	2.8%	19.0%	2.3%	0.6
u.s. gold and metals	3.6%	35.1%	51%	47%	−19.9%	−6.1%	−0.8%	6.0%	39.3%	1.1%	0.5
u.s. health care firms	13.6%	14.0%	36%	64%	−10.5%	−1.0%	1.8%	3.3%	10.3%	11.3%	0.5
u.s. real estate inv. trusts	−0.4%	12.6%	50%	49%	−9.2%	−2.2%	−0.2%	2.0%	9.6%	−1.2%	0.2
u.s. utilities firms	−2.3%	13.9%	57%	43%	−11.9%	−2.5%	−0.7%	2.4%	7.7%	−4.2%	0.4
intl emerging market firms	−4.4%	28.0%	50%	50%	−26.1%	−6.2%	−0.4%	4.5%	15.4%	−9.7%	1.1
intl european firms	1.9%	17.6%	41%	56%	−13.1%	−3.4%	0.3%	3.2%	9.7%	−1.7%	0.7
intl pacific firms	−7.4%	21.3%	59%	40%	−12.1%	−5.3%	−2.0%	4.0%	17.6%	−10.6%	0.7
intl growth firms	−1.6%	17.5%	46%	53%	−13.2%	−3.2%	0.5%	3.1%	9.6%	−5.1%	0.7
intl value firms	−4.2%	18.7%	57%	43%	−15.0%	−3.1%	−0.7%	2.9%	13.0%	−7.5%	0.7
s&p 500 index	4.8%	18.5%	46%	54%	−14.6%	−3.1%	0.5%	5.0%	9.7%	0.0%	1.0
dow jones 30 index	6.3%	18.7%	43%	57%	−15.1%	−3.6%	0.9%	4.0%	10.6%	1.8%	0.9

An important statistic, used later again, is the covariation of investments with the stock market.

The third set of statistics that this chapter described was covariation, which included beta. The last column in the Table 10.4 shows the beta of the rate of return of each investment portfolio with the rate of return on the S&P500 index,

$$\beta_{B,\,S\&P500} = \frac{Cov(\,\tilde{r}_B\;,\;\tilde{r}_{S\&P500}\,)}{Var(\tilde{r}_{S\&P500})}$$

These "market-betas" tell us how much a particular investment portfolio's rate of return covaried with the rate of return on the S&P500. A beta of 1 tells us that the rate of return of a portfolio tended to covary one-to-one with the rate of return in the U.S. stock market. A beta of 0 tells us that the rate of return of a portfolio tended to be unrelated to what happened to the U.S. stock market.

SIDE NOTE

The linear regression by which the beta measure can be obtained is so common that it is called the **market model**, and this particular beta is called the **market beta**. It can be obtained by running the time-series regression

$$\tilde{r}_B = \alpha + \beta \cdot \tilde{r}_{S\&P500}$$

Again, we use a historical beta as estimate for the future beta.

Equities covaried more with the s&p than bonds.

The last column shows that government bonds had practically no covariation with the S&P500. Corporate bonds, energy stocks, precious metals, health care stocks, real estate investment trusts (REITs) and utilities had very mild covariation, indicated by betas around 0.5. The next-most correlated segment are international stocks, having betas of around 0.7. But many other portfolios varied about 1-to-1 with the overall stock market. Note that U.S. growth firms swung even more than 1-to-1 with changes in the stock market: The beta of 1.3 tells us that a 10% increase in the stock market tended to be associated with a 13% increase in the growth firm portfolio. This is typical.

Can you read and understand tables of historical performances now?

For your curiosity, there are two more tables with the same statistics: Table a describes the historical performance of non-U.S. stock markets; and Table b describes the historical performance of the 30 stocks that constitute the Dow-Jones 30 Index. At this point, you can read and interpret the table as well as I can, so enjoy!

SIDE NOTE

Table 10.4 also shows some other statistics, such as the percent of all months that earned a positive rate of return. (Naturally, one minus this percent are the months in which the portfolio had a negative return.) The Table further shows the single worst month, the single best month, the median month (half of all return months were better, half were worse). Finally, although this is beyond what we have covered so far, the alpha (α) in the Table is sometimes interpreted as a risk-adjusted reward measure—the higher the better.

Some other random observations: The table shows no systematic relationship between risk and rate of return over the sample period. However, it is the case that the least risky and least covarying investment strategies (government bonds) provided a very modest, but positive average rate of return. With hindsight, it would have been terrific to invest in U.S. health care stocks: they had the most spectacular return, plus a very modest risk. Naturally, with hindsight, you could have selected the right six numbers for the lottery. So, which numbers can you trust to be indicative of their future equivalent? First and foremost, covariation measures. They tend to be very stable. Next, standard deviations are reasonably stable. Historical means, however, are very untrustworthy as predictors of the future: it is not especially likely that health care firms will continue to outperform other stocks, and that Japanese firms will continue to underperform other stocks.

Solve Now!

All questions refer to Table 10.4.

Q 10.15 Which investment class portfolio would have done best over the sample period? Do you believe this will continue?

Q 10.16 Which investment class portfolio would have done worst over the sample period? Do you believe this will continue?

Q 10.17 Assuming you had held only one asset class, which investment class portfolio was safest during the sample period? Do you believe this will continue?

Q 10.18 Assuming you had held only one asset class, which investment class portfolio was riskiest during the sample period? Do you believe this will continue?

Q 10.19 Which asset class portfolio had the lowest covariation with the S&P500 index? Do you believe this will continue?

Q 10.20 Which asset class portfolio had the highest covariation with the S&P500 index? Do you believe this will continue?

10·4 Summary

The chapter covered the following major points:

- The formulas in this chapter decompose the statistics of a portfolio return in terms of the statistics of its constituent securities' portfolio returns.

  The formulas are merely alternative computations. You can instead write out the time-series of the portfolio's rates of return and compute the portfolio statistics directly from this distribution.

  You shall use these formulas later, because you want to consider portfolios when you vary the weights. The formulas express the overall portfolio statistics in terms of investment weights, which will make it easier to choose the best portfolio.

- For three statistics, you can take investment-weighted averages:
  1. The portfolio expected rate of return is the investment-weighted average of its components' expected rates of return.
  2. The portfolio covariance with anything else is the investment-weighted average of its components' covariance with this anything else.
  3. The portfolio beta with respect to anything else is the investment-weighted average of its components' beta with respect to this anything else.

- The portfolio variance can not be computed as the investment-weighted average of its components' variances. Instead, it is computed as follows:
  1. For each security, square its weights and multiply it by the variance.
  2. For each pair of different securities, multiply two times the first weight times the second weight times the securities' covariance.
  3. Add up all these terms.

  There are other ways to compute the variance. In particular, you can instead compute the historical portfolio rate of return for each time period, and then compute the variance from this univariate time-series. Or you can use the double summation Formula 10.11.

- For a sense of order-of-magnitude, Table 10.4 provides recent return statistics for some common asset-class portfolios. The appendix gives equivalent statistics for the Dow-Jones 30 stocks and for foreign stock markets.

↦ *Table 10.4 on Page 228*

Appendix

Table 10.5: Summary of Portfolio Algebra in the Context of the Chapter Example

Statistic	Notation	Three Input Securities: S&P500	IBM	Sony	Investment-Weighted Average	Formula	Portfolio of 70% IBM, 20% IBM, and 10% Sony
Expected Return	$\mathcal{E}(\tilde{r}_i)$	19.0%	38.8%	90.3%	Yes	$\sum_{i=1}^{N} w_i \cdot \mathcal{E}(\tilde{r}_i)$	$70\% \cdot 19.0\% + 20\% \cdot 38.8\% + 10\% \cdot 90.3\% = 40.9\%$
Covariance, e.g. with x=S&P500	$\sigma_{i,S\&P500}$	0.0362	0.0330	0.0477	Yes	$\sum_{i=1}^{N} w_i \cdot Cov(\tilde{r}_i, \tilde{r}_x)$	$70\% \cdot 0.0362 + 20\% \cdot 0.0330 + 10\% \cdot 0.0477 = 0.03672$
Beta, e.g. with x= S&P500	$\beta_{i,S\&P500}$	1.00	0.91	1.32	Yes	$\sum_{i=1}^{N} w_i \cdot \beta_{(\tilde{r}_i, \tilde{r}_x)}$	$70\% \cdot 1.00 + 20\% \cdot 0.91 + 10\% \cdot 1.32 = 1.01$
Variance	$\sigma_{i,i} = \sigma_i^2$	0.036	0.150	0.815	**No**	$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i \cdot w_j \cdot \sigma_{i,j}$	Requires three variance terms and the three mutual covariance terms. The latter are not provided in this table. For method, see Page 222, and Formulas 10.10 and 10.14. It is 0.0487 here.
Standard Deviation	σ_i	19%	39%	90%	**No**		Squareroot of variance (which is $\sqrt{0.0487} = 41\%$ here).

The goal of this chapter was to explain these portfolio rules.
Know what they mean and how to use them!

A More Historical Statistics

a. Country Fund Rates of Return

Table 10.6: AMEX Country Funds, Based on Monthly Rates of Returns, Annualized, January 1997 to October 2002.

	Annualized Returns				Monthly Returns					Ann.Market-Model	
Asset	Mean	StdDev	%Neg	%Pos	Worst	Q2	Median	Q3	Best	α_i	$\beta_{i,S\&P}$
australia index fund amex	−0.4%	23.1%	47%	49%	−17.5%	−4.2%	0.0%	3.4%	14.5%	−4.1%	0.8
canada index fund amex	5.0%	23.5%	40%	57%	−22.4%	−4.0%	0.7%	5.5%	11.2%	−0.2%	1.1
sweden index fund amex	1.2%	30.3%	46%	53%	−21.2%	−4.9%	0.6%	5.3%	21.7%	−4.4%	1.2
germany index fund amex	3.9%	28.6%	46%	53%	−24.0%	−4.6%	0.9%	5.1%	24.4%	−1.2%	1.1
hong kong index fund amex	−5.7%	36.0%	56%	41%	−28.1%	−7.4%	−1.3%	5.7%	38.4%	−11.7%	1.2
japan index fund amex	−8.0%	24.2%	59%	41%	−16.5%	−5.9%	−1.2%	3.7%	21.1%	−11.4%	0.7
belgium index fund amex	5.9%	26.6%	44%	56%	−19.0%	−2.7%	0.4%	4.0%	37.9%	2.3%	0.8
netherlands index fund amex	0.2%	22.2%	46%	53%	−17.3%	−3.3%	0.3%	3.6%	14.4%	−3.9%	0.9
austria index fund amex	1.2%	21.0%	47%	49%	−19.7%	−4.2%	0.0%	4.4%	11.9%	−1.0%	0.5
spain index fund amex	6.5%	25.0%	44%	53%	−22.6%	−4.1%	0.3%	4.8%	15.5%	2.0%	0.9
france index fund amex	6.2%	23.5%	43%	57%	−15.0%	−2.5%	0.8%	3.9%	16.8%	2.0%	0.9
singapore index fund amex	−7.5%	37.7%	49%	46%	−27.0%	−6.5%	0.0%	3.6%	40.3%	−13.8%	1.3
uk index fund amex	0.2%	15.4%	43%	54%	−11.9%	−3.2%	0.6%	3.1%	8.6%	−3.1%	0.7
mexico index fund amex	11.3%	36.1%	44%	53%	−35.3%	−7.0%	3.0%	9.0%	20.6%	4.7%	1.4
s&p 500 index	4.8%	18.5%	46%	54%	−14.6%	−3.1%	0.5%	5.0%	9.7%	0.0%	1.0

b. Dow-Jones Constituents

Table 10.7: Dow Jones Constituents, Based on Monthly Rates of Returns, Annualized, January 1997 to October 2002.

	Annualized Returns				Monthly Returns					Ann.Market-Model	
Asset	Mean	StdDev	%Neg	%Pos	Worst	Q2	Median	Q3	Best	α_i	$\beta_{i,S\&P}$
alcoa	15.1%	41.3%	49%	51%	−23.9%	−6.2%	0.4%	8.2%	51.1%	8.8%	1.3
american express	17.3%	31.4%	36%	63%	−29.3%	−3.0%	1.7%	8.4%	16.9%	11.3%	1.2
boeing	−2.9%	34.6%	46%	53%	−34.6%	−7.1%	0.6%	6.6%	20.2%	−6.0%	0.7
citigroup	24.9%	37.4%	43%	57%	−34.0%	−5.8%	2.4%	9.0%	25.8%	17.6%	1.5
caterpillar	9.6%	34.0%	46%	54%	−17.4%	−5.9%	0.8%	6.7%	40.8%	5.7%	0.8
du pont	4.4%	29.0%	50%	50%	−17.0%	−6.1%	−0.2%	6.5%	21.7%	0.6%	0.8
disney	0.3%	32.2%	46%	54%	−26.8%	−6.4%	0.4%	5.0%	24.2%	−4.0%	0.9
eastman kodak	−5.5%	34.6%	53%	47%	−34.4%	−6.6%	−0.5%	5.5%	24.2%	−8.4%	0.6
general electric	12.8%	28.4%	50%	49%	−17.7%	−4.1%	−0.2%	5.9%	19.2%	7.6%	1.1
general motors	5.1%	36.7%	50%	49%	−24.1%	−5.0%	−0.5%	5.8%	25.4%	0.1%	1.0
home depot	22.8%	35.1%	41%	59%	−20.6%	−4.1%	2.8%	8.8%	30.2%	16.8%	1.2
honeywell	6.0%	44.5%	44%	56%	−38.4%	−4.1%	0.7%	6.5%	51.0%	0.1%	1.2
hewlett packard	4.0%	48.2%	51%	49%	−32.0%	−9.1%	−1.4%	8.7%	35.4%	−3.1%	1.5
ibm	20.2%	39.6%	50%	50%	−22.6%	−5.8%	−0.6%	7.8%	35.4%	13.2%	1.5
intel	15.8%	53.1%	46%	54%	−44.5%	−9.6%	0.8%	11.4%	33.9%	7.5%	1.7
international paper	6.0%	36.0%	47%	53%	−22.3%	−6.3%	0.5%	5.3%	27.0%	1.6%	0.9
johnson and johnson	19.5%	26.4%	43%	57%	−16.0%	−2.7%	1.5%	6.2%	17.4%	17.0%	0.5
jp morgan	5.9%	42.5%	46%	54%	−30.6%	−6.3%	0.7%	8.2%	32.9%	−1.9%	1.6
coca-cola	3.7%	30.7%	49%	51%	−19.1%	−4.7%	0.1%	6.7%	22.3%	0.9%	0.6
mcdonalds	0.6%	28.8%	44%	56%	−25.7%	−5.7%	1.0%	5.9%	17.7%	−3.1%	0.8
3m	12.8%	26.0%	44%	56%	−15.8%	−3.9%	1.1%	6.0%	25.8%	10.4%	0.5
philip morris	6.9%	33.0%	41%	57%	−23.7%	−4.6%	0.8%	5.3%	24.4%	5.1%	0.4
merck	12.1%	32.0%	49%	51%	−21.7%	−5.6%	0.1%	6.5%	22.8%	9.3%	0.6
microsoft	28.4%	50.0%	47%	51%	−34.4%	−8.7%	0.1%	11.9%	40.8%	20.0%	1.7
proctor and gamble	13.7%	28.4%	41%	59%	−35.4%	−1.8%	1.4%	5.3%	24.7%	12.6%	0.2
sbc communications	8.1%	34.4%	50%	50%	−18.8%	−7.0%	−0.2%	5.7%	29.3%	4.5%	0.8
att	−5.7%	40.3%	54%	46%	−23.8%	−8.5%	−2.6%	6.1%	39.1%	−10.3%	1.0
united technologies	18.1%	34.1%	40%	60%	−32.0%	−3.4%	1.9%	7.0%	24.6%	12.6%	1.1
wal-mart	31.5%	31.1%	33%	66%	−20.8%	−3.2%	3.0%	8.2%	26.4%	27.2%	0.9
exxon	7.1%	18.3%	50%	50%	−10.2%	−3.5%	−0.1%	4.7%	17.7%	4.9%	0.5
dow jones 30 index	6.3%	18.7%	43%	57%	−15.1%	−3.6%	0.9%	4.0%	10.6%	1.8%	0.9
s&p 500 index	4.8%	18.5%	46%	54%	−14.6%	−3.1%	0.5%	5.0%	9.7%	0.0%	1.0

[No keyterm list for statspfios-g.]

End of Chapter Problems

20 "Solve Now" Answers

1.

$$\mathcal{E}(\tilde{r}_P) = \frac{(-0.176) + (-0.290) + \cdots + (-0.265)}{12} = 0.183$$

$$\mathcal{V}ar(\tilde{r}_P) = \frac{(-0.359)^2 + (-0.474)^2 + \cdots + (-0.448)^2}{11} = 0.167$$

$$Cov(\tilde{r}_{S\&P500}, \tilde{r}_P) = \frac{(+0.1620)\cdot(-0.359) + (-0.0565)\cdot(-0.474) + \cdots + (-0.3348)\cdot(-0.448)}{11} = 0.0379$$

$$Cov(\tilde{r}_{IBM}, \tilde{r}_P) = \frac{(-0.3661)\cdot(-0.359)\cdot + (-0.5874)\cdot(-0.474) + \cdots + (-0.5105)\cdot(-0.448)}{11} = 0.1075$$

$$Cov(\tilde{r}_{Sony}, \tilde{r}_P) = \frac{(-0.3448)\cdot(-0.359)\cdot + (-0.2458)\cdot(-0.474) + \cdots + (-0.3228)\cdot(-0.448)}{11} = 0.2862$$

$$Sdv(\tilde{r}_P) = \sqrt{\mathcal{V}ar(\tilde{r}_P)} = \sqrt{16.71\%} = 40.9\%$$

2. There is no error.

3.

$$Cov(\tilde{r}_{IBM}, \tilde{r}_S) = 0.05397, \quad Cov(\tilde{r}_{Sony}, \tilde{r}_S) = 0.6166, \quad Cov(\tilde{r}_{S\&P500}, \tilde{r}_S) = 0.04403$$

4.

$$\beta_{S,S\&P500} = Cov(\tilde{r}_S, \tilde{r}_{S\&P500})/\mathcal{V}ar(\tilde{r}_{S\&P500}) = 0.04403/0.03622 \approx 1.22$$

$$\beta_{S,S\&P500} = w_{IBM}\cdot\beta_{IBM,S\&P500} + w_{Sony}\cdot\beta_{Sony,S\&P500}$$

$$= 25\%\cdot 0.910 + 75\%\cdot 1.317 \approx 1.22$$

5.

$$\mathcal{V}ar(\tilde{r}_P) = Cov(\tilde{r}_S, \tilde{r}_S) = \mathcal{V}ar(\tilde{r}_P) = (25\%)^2\cdot 0.15035 + (75\%)^2\cdot 0.81489$$

$$+2\cdot 25\%\cdot 75\%\cdot 0.02184 = 0.4760$$

6. Do it!

$$\mathcal{E}(\tilde{r}_Q) = \frac{0.0131 + \cdots + (-0.243)}{12} \approx 12.57\%$$

$$Cov(\tilde{r}_Q, \tilde{r}_{S\&P500}) = \frac{(-0.006\cdot 0.1620) + \cdots + (-0.369\cdot -0.3348)}{12} \approx ??.$$

7.

$$\mathcal{E}(\tilde{r}_S) = 25\%\cdot 10.1\% + 35\%\cdot 15.4\% + 40\%\cdot 24.2 \approx 17.59\%$$

$$Cov(\tilde{r}_S, S\&P500) = 25\%\cdot 0.03622 + 35\%\cdot 0.03298 + 40\%\cdot 0.04772 \approx 0.0397$$

$$\beta_{S,S\&P500} = 25\%\cdot 1.00 + 35\%\cdot 0.9104 + 40\%\cdot 1.3172 \approx 1.096$$

8. For the historical computation, compute the returns and their deviations:

Year	1991	1992	1993	1994	1995	1996
Historical Return	−0.04965	−0.14210	+0.25130	+0.15549	+0.21218	+0.31198
Deviation from Mean	−0.22556	−0.31801	+0.07539	−0.02043	+0.03626	+0.13607

Year	1997	1998	1999	2000	2001	2002
Historical Return	+0.36708	+0.25242	+1.29560	−0.30389	−0.02390	−0.21557
Deviation from Mean	+0.19117	+0.07651	+1.11969	−0.47980	−0.19981	−0.39148

because the portfolio mean is 17.59%. The variance is therefore

$$\mathcal{V}ar(\tilde{r}_S) = \frac{0.0509 + 0.1011 + \cdots + 0.1533}{11} \approx 0.1725$$

The alternative calculation is

$$\begin{aligned}\mathcal{V}ar(\tilde{r}_S) &= 0.25^2 \cdot 0.03622 + 0.35^2 \cdot 0.15035 + 0.40^2 \cdot 0.81849 \\ &\quad +2 \cdot 0.25 \cdot 0.35 \cdot 0.03298 \\ &\quad +2 \cdot 0.25 \cdot 0.40 \cdot 0.04772 \\ &\quad +2 \cdot 0.35 \cdot 0.45 \cdot 0.02184 \\ &= 0.1516 + 0.02220 \\ &\approx 0.1738\end{aligned}$$

and the difference is rounding error (of which half is in the Sony variance term).

9. The variance is 0.0487, the standard deviation is 0.2207. (Divide by $N-1$, not by N.)

10. $\mathcal{E}(\tilde{r}_{BF}) = 17.00\%$, $\mathcal{S}dv(\tilde{r}_{BF}) = 35.65\%$

11. $\mathcal{E}(\tilde{r}_{CF}) = 12.74\%$, $\mathcal{S}dv(\tilde{r}_{CF}) = 25.12\%$.

12. $\mathcal{E}(\tilde{r}_{DF}) = 13.00\%$, $\mathcal{S}dv(\tilde{r}_{DF}) = 22.90\%$.

13.

$$\mathcal{E}(\tilde{r}_{EF}) = 10\% \cdot 0.83\% + 10\% \cdot 0.90\% + 80\% \cdot 1.19\% = 1.125\%$$

$$\begin{aligned}\mathcal{V}ar(\tilde{r}_{EF}) &= 10\% \cdot (7.47\%)^2 + 10\% \cdot (8.35\%)^2 + 80\% \cdot (6.29\%)^2 \\ &\quad +2 \cdot 10\% \cdot 10\% \cdot 53.2\% + 2 \cdot 10\% \cdot 80\% \cdot 10.8\% + 2 \cdot 10\% \cdot 80\% \cdot 9.9\% \\ &= 0.0004858 \\ \to \mathcal{S}dv(\tilde{r}_{EF}) &= 6.97\%\end{aligned}$$

14. We want the slope of a line where KO is the X variable. Therefore, the slope is

$$\beta_{PEP,KO} = \frac{\mathcal{C}ov(\tilde{r}_{PEP}, \tilde{r}_{KO})}{\mathcal{V}ar(\tilde{r}_{KO})} = \frac{0.003318}{0.006967} \approx 0.48$$

$$\beta_{PEP,CSG} = \frac{\mathcal{C}ov(\tilde{r}_{PEP}, \tilde{r}_{CSG})}{\mathcal{V}ar(\tilde{r}_{CSG})} = \frac{0.0005}{0.004} \approx 0.13$$

15. Health care firms. Unlikely: You know that historical expected rates of return are not reliable predictors of future expected rates of return.

16. Pacific (Japanese) Firms. Unlikely: You know that historical expected rates of return are not reliable predictors of future expected rates of return.

17. Government Bonds, short term. Likely: Historical standard deviations tend to be good predictors of future standard deviations.

18. Gold and Metals, then Emerging Stock Market Investments. (These are stock markets from developing countries.) Likely: Historical standard deviations tend to be good predictors of future standard deviations.

19. Look at the final column, $\beta_{i,S\&P500}$. Government bonds of all kinds had almost no correlation with the S&P500. Among more risky securities, REITs were almost uncorrelated, too. Likely: historical covariations tend to be good predictors of future covariations.

20. Growth firms. These contain many technology firms. Likely: historical covariations tend to be good predictors of future covariations.

All answers should be treated as suspect. They have only been sketched and have not been checked.