CHAPTER 11[§]

The Principle of Diversification

Eggs and Baskets

This chapter appears in the Survey text only.

HAVING the statistical artillery now in place to describe risk (i.e., the standard deviation), you are ready to abandon the previously maintained assumption of investor risk-neutrality. Henceforth, you will no longer be assumed to be indifferent among investments with the same expected rates of return. Instead, you can now prefer the less risky investment *if* two investment options have the same expected rate of return.

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11·1 What Should You Care About?

What we have assumed about investor preferences?

For the remainder of this book, we are assuming that you care only about the risk and reward of our portfolios, and at one particular point in the future. (You may however reinvest your portfolio at this point to earn more returns.) You care about no other characteristics of your portfolio, or whether a bigger portfolio at this point in time might cause a lower portfolio at the next point in time. What does this mean, and how reasonable are these assumptions?

What non-financial preferences have you assumed away?

First, you are assuming that you do not care about anything other than financial returns. Instead, you could care about whether your portfolio companies invest ethically, e.g., whether a firm in your portfolio produces cigarettes or cancer cures. In real life, few investors care about what their portfolio firms are actually doing. Even if *you* care, you are too small to be able to influence companies one way or the other—and other investors stand ready to purchase any security you may spurn. Aside, if you purchase an ordinary mutual fund, you will hold all sorts of companies—companies whose behavior you may or may not like.

What have you assumed away by looking at the portfolio problem in isolation?

Second, you are assuming that external influences do not matter—you consider your portfolio's outcome by itself without regard for anything else. For example, this means that you do not seek out portfolios that offer higher rates of return if you were to lose your job. *This would not be a bad idea*—you probably should prefer a portfolio with a lower mean (given the same standard deviation), just as long as the better outcomes occur in recessions when you are more likely to lose your job. You should definitely consider such investment strategies—but unfortunately few investors do so in the real world. (If anything, the empirical evidence suggests that many investors seem to do the exact opposite of what they should do if they wanted to ensure themselves against employment risk.) Fortunately, our tools would still work with some modifications if you define your portfolio return to include your labor income.

What return preferences have you assumed away?

Third, you are assuming that risk and reward is all you care about. But it is conceivable that you might care about other portfolio characteristics. For example, the following two portfolios both have a mean return of 20% and a standard deviation of 20%.

Pfio "Symmetric"	with 50% probability, a return of 0%	with 50% probability, a return of +40%
Pfio "Skewed"	with 33% probability, a return of –8%	with 67% probability, a return of +34%

Are you really indifferent between the two? They are not the same. The symmetric portfolio cannot lose money, while the skewed portfolio can. On the other hand, the skewed portfolio has the better return more frequently. By focusing only on mean and standard deviation, you have assumed away any preference between these two portfolios. Few investors in the real world actively invest with an eye towards portfolio return skewness, so ignoring it is acceptable.

What multi-period return processes have you assumed away?

Fourth, you are assuming that you want to maximize your portfolio value at one specific point in time. This could be problematic if, for example, in the symmetric portfolio case it were true that a return of 0% was always later followed by a return of 100%, while a return of +40% was always followed by an unavoidable return of –100% (nuclear war!), then you might not care about the return at the end of the first measurement period. This is so unrealistic that you can ignore this issue for most practical purposes.

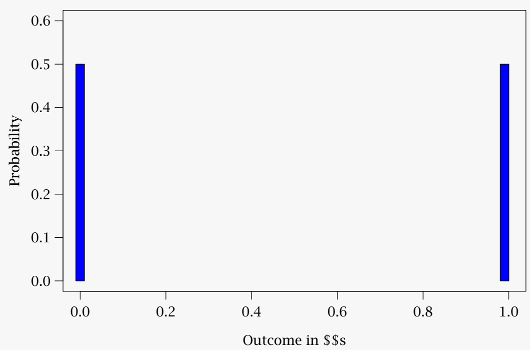
IMPORTANT: The remainder of this book assumes that you care only about the risk (standard deviation) and reward (expected return) of your portfolio.

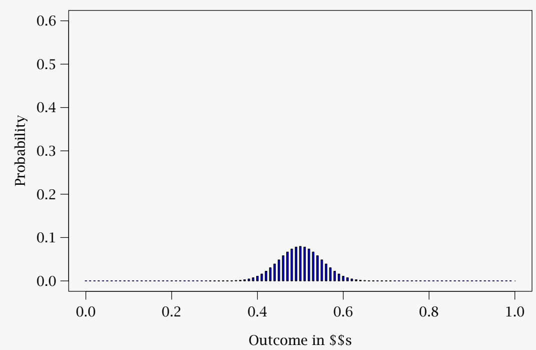
11·2 Diversification: The Informal Way

Don't put all your eggs into one basket (on one bet), if you dislike variance!

Let us assume that you dislike wealth risk. An important method to reduce this risk—and the cornerstone of the area of investments—is **diversification**, which means investing not only in one but in many different assets. We shall expound on it in great detail, but it can be explained with a simple example. Compare two bets. The first bet depends on the outcome of a single coin toss. If heads, the bet pays off \$1 (if tails, the bet pays off nothing). The second bet depends on the outcome of 100 coin tosses. Each coin toss, if heads, pays off $^1/_{100}$ of a dollar, zero otherwise. The expected outcome of either bet is 50 cents. But, as Figure 11.1 shows, the standard deviation of the payoff—the risk—of the latter bet is much lower.

Figure 11.1: Payoffs Under Two Bets With Equal Means.





The left figure is the distribution of payoff where you bet \$1 on heads for 1 coin throw. The right figure is the distribution of payoff where you bet \$0.01 on heads for each of 100 coin throws.

What about gambling?

We are assuming that investors dislike risk. Putting money into many different bets, rather than one big bet, accomplishes this goal. But, is this a good strategy on the roulette table, too? Should you bet your entire money on red (one big bet only), or should you bet it one dollar at a time on red? From a purely financial perspective, the answer is that the single bet is better: if you bet one dollar at a time, you are indeed likely to have a lower variance of payoffs around your expected rate of return. Unfortunately, in roulette, your expected rate of return is *negative*. The casino would be perfectly happy to have you pay up your (negative) expected rate of return *without any risk* for each roll of the ball. Indeed, if your strategy is to gamble until you either are bankrupt or have doubled your money, you are more likely not to go bankrupt if you make fewer but bigger bets.

Anecdote: Ancient Wisdom: How Cato (the Elder) diversified.

Plutarch, a famous Historian who lived in the first century AD, notes Cato's famous statement that, if people wished to obtain money for shipping business [from him], they should form a large association and when the association had fifty members and as many ships, he would take one share in the company.

11·3 Diversification: The Formal Way

11·3.A. Uncorrelated Securities

The base case: independent security returns.

↦ Unknown 8·3 on Page 176

Recall from Section 8·3 that a portfolio's return, r_P, is

$$r_P \equiv \sum_{i=1}^{N} w_i \cdot r_i$$

where P is the overall portfolio, w_i is the investment weight (proportion) in asset i, and i is a counter that enumerates all assets from 1 to N. If you do not yet know the return outcomes, then your returns are random variables, so

$$\tilde{r}_P \equiv \sum_{i=1}^{N} w_i \cdot \tilde{r}_i$$

It is now time to put the laws of expectations and standard deviations (from Section 10·2) to good use. To illustrate how diversification works, make the following admittedly unrealistic assumptions:

1. All securities offer the same expected rate of return (mean):

   $$\mathcal{E}(\tilde{r}_i) = 5\% \quad \text{for all } i \;;$$

2. All securities have the same risk (standard deviation of return):

   $$Sdv(\tilde{r}_i) \equiv \sigma(\tilde{r}_i) \equiv \sigma_i = 40\% \quad \text{for all } i$$

   which is roughly the annual rate of return standard deviation for a typical U.S. stock;
3. All securities have rates of return that are independent from one another. Independence implies that security returns have zero covariation with one another, so $Cov(\tilde{r}_i, \tilde{r}_j) = 0$ for any two securities i and j—just as long as i is not j:

   $$Cov(\tilde{r}_i, \tilde{r}_j) = \sigma_{i,j} = 0 \qquad \text{for all different } i \text{ and } j$$

   (This last assumption is the most unrealistic of the three.)

The portfolio definition.

What are the risk and return characteristics of the portfolio, $\tilde{r}_P$, if it contains N securities? For an equal-weighted portfolio with N securities, each investment weight is $1/N$, so the portfolio rate of return is

$$\tilde{r}_P = \sum_{i=1}^{N} w_i \cdot \tilde{r}_i = \sum_{i=1}^{N} (1/N) \cdot \tilde{r}_i = 1/N \cdot \tilde{r}_1 + 1/N \cdot \tilde{r}_2 + \cdots + 1/N \cdot \tilde{r}_N$$

The Expected Rate of Return.

Let's start with one security. In this case, $r_P \equiv r_1$, so

$$\mathcal{E}(\tilde{r}_P) = \mathcal{E}(\tilde{r}_1) = 5\% \;, \quad Sdv(\tilde{r}_P) = Sdv(\tilde{r}_1) = \sqrt{Var(\tilde{r}_1)} = 40\%$$

Two securities now. The portfolio consists of a 50-50 investment in securities 1 and 2:

$$\tilde{r}_P = 1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2$$

In this portfolio, it is easy to see that the average expected rate of return on a portfolio is the average of the expected rates of return on its components:

$$\begin{aligned}\mathcal{E}(\tilde{r}_P) &= \mathcal{E}(1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2) \\ &= 1/2 \cdot \mathcal{E}(\tilde{r}_1) + 1/2 \cdot \mathcal{E}(\tilde{r}_2) \\ &= 1/2 \cdot 5\% + 1/2 \cdot 5\% = 5\%\end{aligned}$$

More generally, it is not a big surprise that the rate of return is 5%, no matter how many securities enter the portfolio:

$$\begin{aligned}\mathcal{E}(\tilde{r}_P) &= \sum_{i=1}^{N} \mathcal{E}(1/N \cdot \tilde{r}_i) = \sum_{i=1}^{N} 1/N \cdot \mathcal{E}(\tilde{r}_i) \\ &= \sum_{i=1}^{N} 1/N \cdot 5\% = 5\%\end{aligned}$$

The base case: Variance and Standard Deviation are lower for 2 securities.

It is when you turn to the portfolio risk characteristics that it becomes interesting. The variance and standard deviation of the rate of return on the portfolio P are more interesting. Begin with two securities:

$$\begin{aligned}\mathcal{V}ar(\tilde{r}_P) &= \mathcal{V}ar(1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2) \\ &= (1/2)^2 \cdot \mathcal{V}ar(\tilde{r}_1) + (1/2)^2 \cdot \mathcal{V}ar(\tilde{r}_2) + 2 \cdot (1/2) \cdot (1/2) \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_2) \\ &= 1/4 \cdot 0.16 + 1/4 \cdot 0.16 + 2 \cdot 1/2 \cdot 1/2 \cdot 0 \\ &= 1/4 \cdot 0.16 + 1/4 \cdot 0.16 \\ &= 1/2 \cdot 0.16 \\ \Longleftrightarrow \mathcal{S}dv(\tilde{r}_P) &= \sqrt{\mathcal{V}ar(\tilde{r}_P)} = \sqrt{1/2 \cdot 0.16} = 70.7\% \cdot 40\% = 28.3\%\end{aligned} \tag{11.1}$$

You could drop out the covariance term, because we have assumed security returns to be independent. Pay close attention to the final line: the portfolio of one security had a risk of 40%. The portfolio of two securities has the lower risk of 28.3%. This is important—diversification at work!

SIDE NOTE

If you find it easier to understand the formula if you see some data, below are two sample series that are consistent with our assumptions: each has 5% mean and 40% standard deviation, and they have zero mutual covariance. The final column is the rate of return on the portfolio P that invests half in each security, thus appropriately rebalanced each year, of course . You can confirm that the standard deviation of this portfolio is indeed the same 28.284% that you have just computed via the formula.

Year	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_P$	Year	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_P$
1980	25.420	-11.476	6.972	1985	21.268	-35.560	-7.146
1981	39.376	62.064	50.720	1986	-45.524	30.620	-7.452
1982	41.464	-19.356	11.054	1987	42.976	59.616	51.296
1983	30.500	-41.256	-5.378	1988	-10.324	30.360	10.018
1984	-67.552	15.772	-25.890	1989	-27.604	-40.784	-34.194
				$\mathcal{E}$	5.000	5.000	5.000
				$\mathcal{S}dv$	40.000	40.000	28.284

Variance and Standard Deviation for N securities.

Now, compute the standard deviation for an arbitrary number of securities in the portfolio. Recall the variance formula,

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= \mathcal{V}ar\left(\sum_{i=1}^{N} c_i \cdot \tilde{r}_i\right) = \sum_{j=1}^{N}\left\{\sum_{k=1}^{N}\left[c_j \cdot c_k \cdot Cov(\tilde{r}_j, \tilde{r}_k)\right]\right\} \\
& w_1^2 \cdot \mathcal{V}ar(\tilde{r}_1) + w_2^2 \cdot \mathcal{V}ar(\tilde{r}_2) + \cdots + w_N^2 \cdot \mathcal{V}ar(\tilde{r}_N) \\
&+ 2 \cdot w_1 \cdot w_2 \cdot Cov(\tilde{r}_1, \tilde{r}_2) + 2 \cdot w_1 \cdot w_3 \cdot Cov(\tilde{r}_1, \tilde{r}_3) + \cdots + 2 \cdot w_1 \cdot w_N \cdot Cov(\tilde{r}_1, \tilde{r}_N) \\
&+ 2 \cdot w_2 \cdot w_1 \cdot Cov(\tilde{r}_2, \tilde{r}_1) + 2 \cdot w_2 \cdot w_3 \cdot Cov(\tilde{r}_2, \tilde{r}_3) + \cdots + 2 \cdot w_2 \cdot w_N \cdot Cov(\tilde{r}_2, \tilde{r}_N) \\
&+ \vdots + \ddots + \ddots + \vdots \\
&+ 2 \cdot w_N \cdot w_1 \cdot Cov(\tilde{r}_N, \tilde{r}_1) + 2 \cdot w_N \cdot w_3 \cdot Cov(\tilde{r}_N, \tilde{r}_3) + \cdots + 2 \cdot w_N \cdot w_{N-1} \cdot Cov(\tilde{r}_N, \tilde{r}_{N-1})
\end{aligned} \tag{11.2}$$

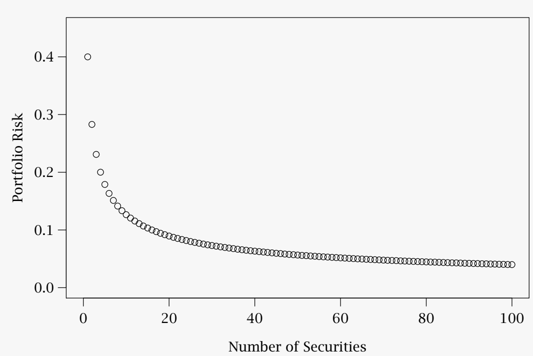
In our example, all the covariance terms are zero, all the weights are $1/N$, and all individual security variances are the same, so this is

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_1) + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_2) + \cdots + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_N) \\
&= (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) + \cdots + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) \\
&= N \cdot (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) = (1/N) \cdot \mathcal{V}ar(\tilde{r}_i) \\
\Longleftrightarrow \quad Sdv(\tilde{r}_P) &= \sqrt{1/N \cdot \mathcal{V}ar(\tilde{r}_i)} = \sqrt{1/N} \cdot Sdv(\tilde{r}_i) = \sqrt{1/N} \cdot 40\%
\end{aligned}$$

Reflect on the effectiveness of diversification: a lot for the first few additions, then less and less.

This formula states that for 1 security, the risk of the portfolio is 40%; for 2 securities, it is 28.3% (as also computed in Formula 11.1); for 4 securities, it is $\sqrt{1/4} \cdot 40\% = 20\%$; for 16 securities, it is 10%; for 100 securities, it is 4%, and for 10,000 securities, is 0.4%. In a portfolio of infinitely many securities, the risk of the portfolio gradually disappears. In other words, you would practically be certain to earn the expected rate of return (here 5%). Now take a look at the risk decline in Figure 11.2 to see how more securities help to reduce risk. The square root function on N declines steeply for the first few securities, but then progressively less so for subsequent securities. Going from 1 to 4 securities reduces the risk by 50%. The next 5 securities (going from 4 to 9 securities) only reduce the risk by 17% (from $\sqrt{1/4} = 50\%$ to $\sqrt{1/9} = 33\%$). To drop the risk from 50% to 25% requires 12 extra securities; to drop the risk from 50% to 10% requires 96 extra securities. In other words, *if* security returns are independent, diversification works really well in the beginning, but less and less as more securities are added. It is important to have, say, a dozen independent securities in the portfolio, which drops the portfolio risk by two-thirds; additional diversification through purchasing more securities is nice, but it is not as important, in relative terms, as these first dozen securities.

Figure 11.2: Diversification if security returns were independent.



11·3.B. Correlated Securities

When does diversification fail? Recall that the maximum possible correlation of 1 implies that

Diversification fails when security returns are perfectly positively correlated.

↦ Max Correlation: 9·4.B on Page 201

$$Correlation(\tilde{X},\tilde{Y}) = +1 \iff Cov(\tilde{X},\tilde{Y}) = Sdv(\tilde{X}) \cdot Sdv(\tilde{Y})$$

Therefore, the covariance of two perfectly correlated securities' rates of returns (in our example) is

$$Cov(\tilde{r}_i,\tilde{r}_j) = 40\% \cdot 40\% = 0.16$$
$$Cov(\tilde{r}_i,\tilde{r}_j) = Sdv(\tilde{r}_i) \cdot Sdv(\tilde{r}_j)$$

The variance of a portfolio of two such stocks is

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= \mathcal{V}ar(1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2) \\
&= 1/2^2 \cdot \mathcal{V}ar(\tilde{r}_1) + 1/2^2 \cdot \mathcal{V}ar(\tilde{r}_2) + 2 \cdot 1/2 \cdot 1/2 \cdot Cov(\tilde{r}_1,\tilde{r}_2) \\
&= 1/4 \cdot 0.16 + 1/4 \cdot 0.16 + 2 \cdot 1/2 \cdot 1/2 \cdot 0.16 \\
&= 1/2 \cdot 0.16 + 1/2 \cdot 0.16 = 0.16 \\
Sdv(\tilde{r}_P) &= \sqrt{\mathcal{V}ar(\tilde{r}_P)} = 40\%
\end{aligned}$$

In other words, when two securities are perfectly correlated, diversification does not reduce portfolio risk. It should come as no surprise that you cannot reduce the risk of a portfolio of one PepsiCo share by buying another PepsiCo share. (The returns of PepsiCo shares are perfectly correlated.) In general, the smaller or even more negative the covariance term, the better diversification works.

IMPORTANT:

- Diversification fails when underlying securities are perfectly positively correlated.
- Diversification reduces portfolio risk better when its underlying securities are less correlated.
- Diversification works perfectly (reducing portfolio risk to zero) when underlying securities are perfectly negatively correlated.

Question 11.3 at the end of this section asks you to prove the last point.

11·3.C. Measures of Contribution Diversification: Covariance, Correlation, or Beta?

It seems that diversification works better when the covariation between investment securities is smaller. The correct measure of overall risk remains, of course, the standard deviation of the portfolio's rate of return. But, the question you are now interested in is "What is the best measure of the contribution of an individual security to the risk of a portfolio?" You need a measure of the risk contribution of just one security inside the portfolio to the overall portfolio risk.

Covariance, Correlation, and Beta are multiple possible measures of covariation.

Let's assume that you already hold a portfolio Y, and you are now considering adding "a little bit" of security i. How much does this new security help or hurt your portfolio through diversification? Because you are adding fairly little of this security, it is a reasonable approximation to assume that the rest of the portfolio remains as it was (even though the new security really becomes

Candidates to measure how a new security helps portfolio diversification.

part of portfolio Y and thereby changes Y). The three candidates to measure how correlated the new investment opportunity i is with the rest of your portfolio Y are

$$\text{The Covariance:} \quad Cov(\tilde{r}_i, \tilde{r}_Y) \quad \text{(Uninterpretable)}$$

$$\text{The Correlation:} \quad \frac{Cov(\tilde{r}_i, \tilde{r}_Y)}{Sdv(\tilde{r}_i) \cdot Sdv(\tilde{r}_Y)} \quad \text{(Interpretable)}$$

$$\text{The Portfolio Beta:} \quad \frac{Cov(\tilde{r}_i, \tilde{r}_Y)}{Var(\tilde{r}_Y)} \quad \text{(Interpretable)}$$

↦ *Covariance: Formula 11.1 on Page 243*

Although all three candidates share the same sign, each measure has its own unique advantage. The covariance is used directly in the portfolio formula, but its value is difficult to interpret. The correlation is easiest to interpret, because it lies between −1 and +1. However, its real problem as a measure of risk for a new security (which you want to add to your portfolio) is that it ignores a security's relative variability.

Debunking correlation: it is not a good risk measure.

This last fact merits an explanation. Consider two securities that both have perfect correlation with your portfolio. In our example, there are only two equally likely possible outcomes:

	Your Portfolio	Security A	Security B
Outcome 1	+24%	+12%	+200%
Outcome 2	-12%	-6%	-100%
$\mathcal{E}(\tilde{r})$	+6%	+3%	50%
$Sdv(\tilde{r})$	+18%	+9%	150%

Now assume you had \$75 in your portfolio, but you are adding \$25 of either A or B. Therefore, your new combined portfolio rate of return would be

	Your Portfolio Y Plus Security A	Your Portfolio Plus Security B
Case 1	$75\% \cdot (+24\%) + 25\% \cdot (+12\%) = 21.0\%$	$75\% \cdot (+24\%) + 25\% \cdot (+200\%) = 68\%$
Case 2	$75\% \cdot (-12\%) + 25\% \cdot (-6\%) = -10.5\%$	$75\% \cdot (-12\%) + 25\% \cdot (-100\%) = -34\%$
$\mathcal{E}(\tilde{r}_P)$	5.25%	51.00%
$Sdv(\tilde{r}_P)$	15.75%	153.00%

Adding stock B causes your portfolio risk to go up more. It adds more portfolio risk. Yet, the correlation between your portfolio and either security A or security B was the same. The correlation would not have told you that B is the riskier add-on.

But the beta of a stock with your portfolio still works as a good risk measure!

In contrast to correlation, our third candidate for measuring risk contribution tells you the right thing. The beta of security A with respect to your portfolio Y is 0.5 (which you can compute either with the formula $Cov(\tilde{r}_A, \tilde{r}_Y)/Var(\tilde{r}_Y)$, or by recognizing that A is always one-half of Y); the beta of security B with respect to your portfolio Y is 8.33. Therefore, beta tells you that adding security B would increase your overall portfolio risk more than adding security A. Unlike correlation, beta takes into account the *scale* of investments.

A concrete example that shows that correlation is not a great diversification measure.

You can also find this scale problem within the context of our earlier three-investments scenario with the annual returns of the S&P500, IBM, and Sony. On Page 204, you found that, compared to IBM, Sony had a higher beta with the market, but a lower correlation. Does a portfolio I consisting of one-half S&P500 and one-half IBM have more or less risk than a portfolio S consisting of one-half S&P500 and one-half Sony?

$$Var(\tilde{r}_I) = (1/2)^2 \cdot 3.62\% + (1/2)^2 \cdot 15.03\% + 2 \cdot 1/2 \cdot 1/2 \cdot 3.30\% = 5.075\%$$

$$Var(\tilde{r}_S) = (1/2)^2 \cdot 3.62\% + (1/2)^2 \cdot 81.49\% + 2 \cdot 1/2 \cdot 1/2 \cdot 4.77\% = 21.874\%$$

Even though Sony has lower correlation with the S&P500 than IBM, Sony's higher variance negates this advantage: the S portfolio is riskier than the I portfolio. The covariances reflect this accurately: the covariance of the S&P500 with Sony is higher than the covariance of the S&P500 with IBM. Beta preserves this ordering, too, because both covariances are divided by the same variance (the variance of $\tilde{r}_{S\&P500}$). It is only the correlation that would have misleadingly indicated that Sony would have been the better diversifier. This, then, is our main insight:

IMPORTANT: The beta of any security i with respect to our portfolio (called $\beta_{i,Y}$) is a measure of the security's risk contribution to the portfolio, because it properly takes scale into account. The lower the beta of security i with respect to portfolio Y, the better security i works at diversifying portfolio Y's overall risk.

The other two measures.

The covariance would have worked equally well, but it is more difficult to intuitively interpret. Correlation is not suitable as a quantitative measure, because it fails to recognize investment scales.

Beta has a nice slope interpretation, too.

The remainder of this section just provides some additional intuition: Beta has a nice slope interpretation. If you graph the rate of return on your overall portfolio Y (e.g., the S&P500) on the x axis, and the rate of return on the new security i (e.g., Sony) on the y axis, then beta represents the slope of a line that helps you predict how the rate of return of security i will turn out *if* you know how the rate of return on your portfolio turned out.

Beta can be visually judged by graphing the rate of return on a security against the rate of return on the overall portfolio. It is the slope of the line.

Figure 11.3 shows three securities with very different betas with respect to your portfolio. In graph (c), security POS has a positive beta with respect to your portfolio. You would therefore expect security POS to not help you much in your quest to diversify: when your portfolio Y does better, so does U. In the figure, U has a beta with respect to your portfolio Y of 3. This indicates that if Y were to ear an additional 5% rate of return, the rate of return on the security POS would be expected to change by an additional 15%. More importantly, if your portfolio Y hits hard times, asset POS would be hit even harder! You would be better off purchasing only a very, very small amount of such a security; otherwise, you could quickly end up with a portfolio that looks more like POS than like Y. In contrast, in graph (a), you see that if you purchased security NEG would expect this NEG to help you considerably in diversification: when your Y does worse, NEG does better! With a beta of −3, the security NEG tends to go up by an additional 15% when the rest of your portfolio Y goes down by an additional 5%. Therefore, NEG provides excellent "insurance" against downturns in Y. Finally, in graph (b), the security ZR has a zero beta, which is the case when ZR 's rates of return are independent of P's rates of return. You already know that securities with no correlation can help you quite nicely in helping diversify your portfolio risk.

SIDE NOTE

The intercept in Figure 11.3 is sometimes called the "alpha," and it can measure how much expected rate of return the security is likely to offer, holding its extra risk constant. For example, if stocks i and j have lines as follows

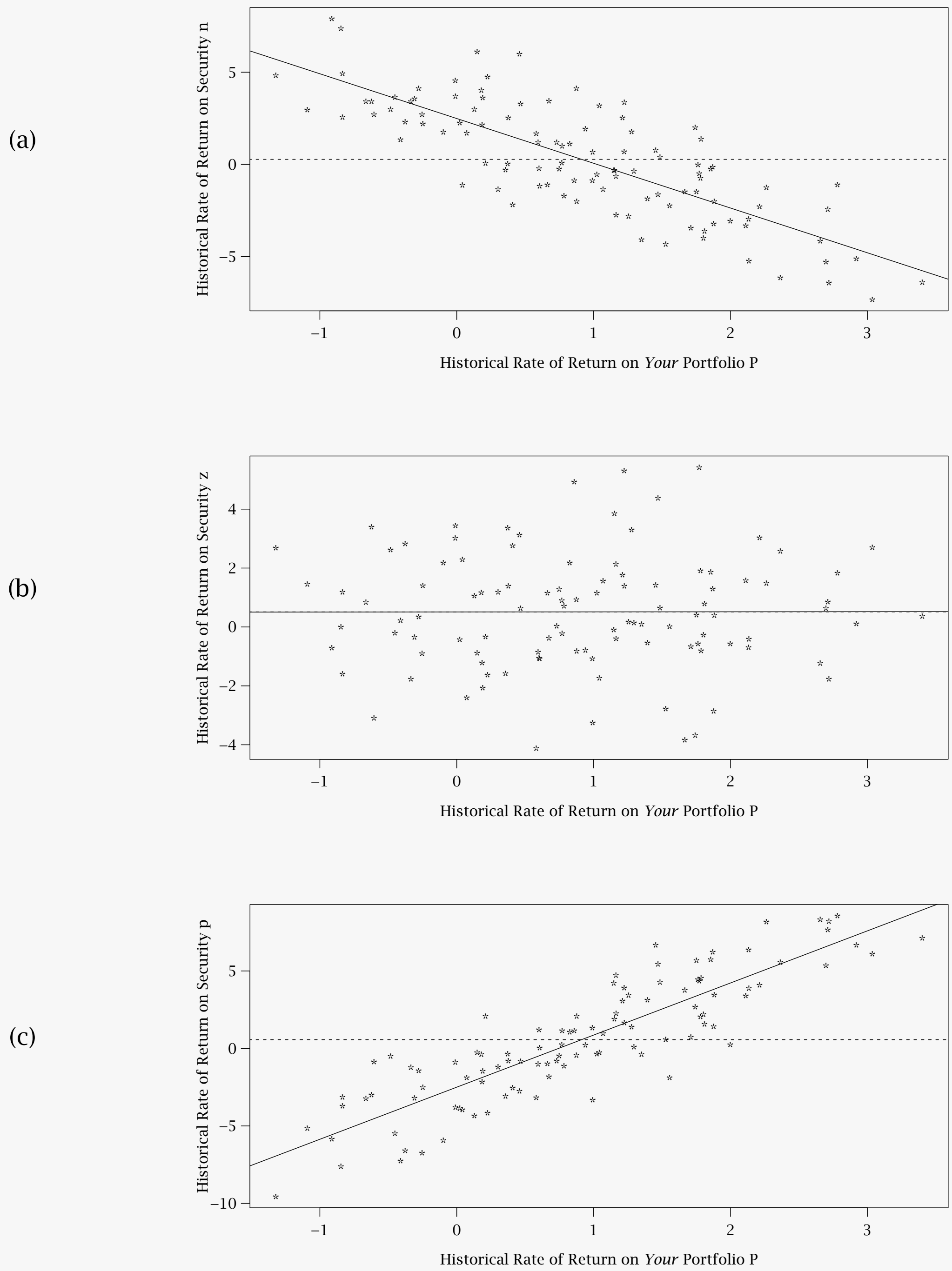
$$\mathcal{E}(\tilde{r}_i) \approx +15\% + 1.5 \cdot \mathcal{E}(\tilde{r}_P)$$

$$\mathcal{E}(\tilde{r}_j) \approx -10\% + 1.5 \cdot \mathcal{E}(\tilde{r}_P)$$

$$\mathcal{E}(\tilde{r}_k) \approx \alpha_k + \beta_{k,P} \cdot \mathcal{E}(\tilde{r}_P)$$

the α of stock i is 15%, while the α of stock j is −10%. It appears that stock i offers a holder of portfolio P a lot of positive return compared to j, holding its exposure to portfolio P constant. Naturally, you would like stocks with high alphas and low betas—but such opportunities are difficult to find, especially if you already hold a reasonably good (well-diversified) portfolio.

Figure 11.3: Three Different Beta Measures of Risk for Security i.



The historical betas of these three securities, n, z, and p, with respect to your portfolio P are based on returns whose true betas with respect to your portfolio P are −3, 0, and +3 in the three graphs, respectively. Historical data is indicative, but not perfect in telling you the true beta.

Solve Now!

Q 11.1 Assume that every security has a mean of 12%, and a standard deviation of 30%. Further, assume that each security has no covariation with any security. What are the risk and reward of a portfolio of N stocks that invests equal amount in each security?

Q 11.2 Assume that every security has a mean of 12%, and a standard deviation of 30%. Further, assume that each security has 0.0025 covariation with any security. What are the risk and reward of a portfolio of N stocks that invests equal amount in each security? If you find this difficult, solve this for 2 stocks. Can you guess what the risk is for a portfolio of infinitely many such stocks?

Q 11.3 Compute the variance of a two-stock portfolio if the two securities are not perfectly positively, but perfectly negatively, correlated.

Q 11.4 You own a $1,000 portfolio P, whose expected rate of return has a mean of 10% and a standard deviation of 20%. You are considering buying a security Q that has a mean of 15% and a standard deviation of 50%. The correlation between the rates of return on P and Q is 20%.

(a) What is the covariance between the rate of return of P and Q?

(b) What is the beta between the rate of return on P and Q?

(c) Consider purchasing $1,000 in Q. What would the portfolio risk be?

A new security, named N has appeared. It has a mean of 150% and a standard deviation of 500%, and the same 20% correlation with P. (Sidenote: Such a security could be created by a fund that borrows in order to purchase more than 100% in Q.)

(d) What is the covariance between the rate of return of P and N?

(e) What is the beta between the rate of return on P and N?

(f) Consider purchasing $1,000 in N. What would the portfolio risk be?

(g) Q and N have equal correlation with portfolio P. Does it follow that they would both be equal risk-contributors, if added to the portfolio?

11·4 Does Diversification Work in the Real World?

You now understand the theory. But can you make it work in the real world?

11·4.A. Diversification Among The Dow-Jones 30 Stocks

Diversification depends on the specific portfolio.

To see whether portfolio diversification works in the real world, you should look at some specific securities. Clearly, the degree to which diversification works must depend on the weights of the specific securities you look at within the context of your portfolio.

Anecdote: Value-At-Risk (VAR)

The latest in risk measurement techniques among banks and other financial institutions—and a great step forward if executed correctly—is **VaR** (**Value-at-Risk**). It often replaces older risk-scoring systems, in which (for example) all commercial loans received one score, government loans another, etc. The goal of Value at Risk is to compute the risk (standard deviation) when all investment (loans) are evaluated in a portfolio framework. Value at Risk can come to very different conclusions than these older systems, especially if payoffs to loans are very negatively or very positively correlated.

The risk of investing in a Dow-Jones 30 Stock.

↦ DJ30 Section 8·3.C on Page 179

You can get an intuitive feel for the effectiveness of diversification in the U.S. stock markets if you look at the 30 stocks constituting the Dow-Jones 30. The Dow-Jones company has chosen 30 stocks in its Dow-Jones 30 Index that avoid industry and firm-type concentration. However, the Dow-Jones contains only very large firms, both in market capitalization and sales.

The average stock in the Dow-Jones has a risk of about 30% to 40%; the overall portfolio has a risk just about half.

Table 11.1 shows the risks (standard deviations of the rate of returns) for the 30 Dow-Jones stocks, measured either from January 1997 to October 2002 or from January 1994 to October 2002. Their risks ranged from about 18% to 50% (annualized); the typical stock's risk averaged about 30% from 1994 to 2002, and 35% from 1997 to 2002. Just investing in one randomly chosen stock is fairly risky. However, when you compute the risk of the index itself, the index's risk was only 16.5% from 1994 to 20002 and 18.7% from 1997 to 2002. Naturally, this is still pretty risky. But the Dow-Jones index risk is only about half the risk of its average stock.

Inferring correlations.

Are the returns of these 30 stocks positively correlated? Note that 18% is only about half of the 35% risk that would obtain if all the securities' returns were perfectly positively correlated. Diversification works: the correlation among the Dow-Jones 30 stock returns is not close to +1. However, the correlation among the Dow-30 stocks is also not zero. Let us do some back-of-the-envelope calculations. If the Dow-Jones 30 Index were an equal-weighted portfolio (it is not!) of uncorrelated securities (it is not!), you would have expected it to have a risk of about $\sqrt{1/30} \cdot 35\% \approx 6.4\%$ per year. Instead, the risk of the Dow-Jones portfolio is 2.8 times as high at 18% per year. In sum, this evidence suggests that these 30 stocks move together, i.e. that they tend to have mutual positive correlations. This reduces the effectiveness of diversification among them. This is actually a broader effect. When the U.S. stock market does well, most stocks do well at the same time (and vice-versa).

IMPORTANT: The Dow-Jones 30 Market Index Portfolio has a risk of about 15% to 20% per year. (Broader U.S. stock market indexes, like the S&P500 Index, tend to have slightly lower risks.)
The 30 component stocks in the Dow-Jones 30 Index are mutually positively correlated, which limits the effectiveness of diversification—but not so much as to render diversification useless.

Make sure not to buy just tech stocks, or just growth stocks, or ...

The square root in the portfolio standard deviation formula suggests that most of the diversification typically comes from the first 10 to 50 stocks. Therefore, it is more important to be suitably diversified across different types of stocks (to avoid mutual positive covariances) than it is to add every single possible stock to a portfolio. If you holds the Dow-Jones 30 stocks if you want to further diversify using U.S. stocks, you should consider adding small or high-growth firm stocks to your portfolio (or, better, invest in a mutual fund that itself holds on many small firms).

A short digression: covariance matters more than variance!

↦ Unknown 11.2 on Page 244

What matters more in determining portfolio variance: covariances or variances? The number of covariance terms in the portfolio risk formula 11.2 increases roughly with the square of the number of securities—by $N^2 - N$ to be exact. The number of variances increases lineary—by N. For example, for 100 securities, there are 100 variance terms and 9,900 covariance terms (4,450 if you do not want to double-count the same pairwise covariance). It should come as no surprise that as the number of securities becomes large, the risk of a portfolio is determined more by the covariance terms than by the variance terms.

A short digression: covariance matters more than variance!

The Dow-Jones Index is not alone in benefiting from the effects of diversification. Academic research has shown that if you look at the average stock on the NYSE, about 75% of its risk can be diversified away (i.e., disappears!), while the remaining 25% of its risk cannot be diversified away. Put differently, undiversifiable co-movements among all stocks in the stock market are responsible for about one-quarter of the typical stocks' return variance; three-quarters are idiosyncratic day-to-day fluctuations, which average themselves away if you hold a highly diversified stock market index like portfolio.

Table 11.1: Risk and Reward for Dow Jones Constituents, Based on Monthly Rates of Returns, Then Annualized.

	about 10 years 1994/01-2002/10		about 5 years 1997/01-2002/10	
Asset	Mean	StdDev	Mean	StdDev
alcoa	18.4%	36.0%	15.1%	41.3%
american express	21.4%	27.9%	17.3%	31.4%
boeing	8.9%	30.1%	−2.9%	34.6%
citigroup	27.7%	33.6%	24.9%	37.4%
caterpillar	14.0%	31.2%	9.6%	34.0%
du pont	12.2%	26.2%	4.4%	29.0%
disney	6.7%	28.8%	0.3%	32.2%
eastman kodak	4.5%	30.0%	−5.5%	34.6%
general electric	17.1%	25.1%	12.8%	28.4%
general motors	5.2%	32.6%	5.1%	36.7%
home depot	18.5%	31.3%	22.8%	35.1%
honeywell	11.2%	37.9%	6.0%	44.5%
hewlett packard	14.8%	42.9%	4.0%	48.2%
ibm	25.8%	36.0%	20.2%	39.6%
intel	28.7%	46.7%	15.8%	53.1%
international paper	7.5%	31.6%	6.0%	36.0%
johnson and johnson	23.2%	23.9%	19.5%	26.4%
jp morgan	15.0%	36.8%	5.9%	42.5%
coca-cola	13.1%	26.5%	3.7%	30.7%
mcdonalds	6.2%	25.6%	0.6%	28.8%
3m	14.8%	23.2%	12.8%	26.0%
philip morris	13.5%	29.9%	6.9%	33.0%
merck	19.2%	28.6%	12.1%	32.0%
microsoft	35.8%	42.6%	28.4%	50.0%
proctor and gamble	16.7%	24.7%	13.7%	28.4%
sbc communications	9.4%	28.9%	8.1%	34.4%
att	−4.4%	36.5%	−5.7%	40.3%
united technologies	21.9%	29.2%	18.1%	34.1%
wal-mart	20.6%	28.8%	31.5%	31.1%
exxon	10.0%	16.8%	7.1%	18.3%
Average	15.3%	31.0%	10.6%	35.0%
Typical (Median)	14.8%	29.9%	8.1%	34.1%
dow jones 30 index	10.5%	16.5%	6.3%	18.7%

For comparison, the S&P500 index had a mean of 8.6% (risk of 16.1%) from 1994–2002, and a mean of 4.8% (risk of 18.5%) from 1997—2002.

SIDE N

In Section a on Page 205, I claimed that historical standard deviations of rates of return tend to be relatively stable. This table shows that, although the riskiness of firms does change over time (it is different over 5 years and 10 years), it does change only slowly, even for the individual Dow-30 stocks. (There would be even more stability if you considered asset class portfolios instead of just stocks.) This stability gives us confidence in using historical risk measures (e.g., standard deviations) as estimates of future risk.

11·4.B. Mutual Funds

There are too many stocks for you to buy them all.

Diversification clearly reduces risk, but it can also be expensive to accomplish. How can you purchase 500 securities with a $50,000 portfolio? The transaction costs of purchasing $100 in each security would be prohibitive. With about 10,000 publicly traded equities in the U.S. stock market, even purchasing just $1,000 in every stock traded would require $10 million, well beyond the financial capabilities of most retail investors.

Mutual funds are investment vehicles to accomplish diversification.

Mutual funds, already mentioned in Section 8·3.B, come to the rescue. As already described in Chapter 8, a mutual fund is like a firm that consists of nothing but holdings in other assets, usually financial assets. In a sense, a mutual fund is a large portfolio that can be purchased as a bundle. However, there is one small catch—isn't there always? On the one hand, investing in a mutual fund rather than in its individual underlying assets can reduce your transaction costs, ranging from the time necessary to research stocks and initiate transactions, to the direct trading costs (the commission and bid-ask spread). But, on the other hand, mutual funds often charge a variety of fees and may force you to realize taxable gains in a year when they would rather not.

11·4.C. Alternative Assets

Other Assets Are Equally or More Important.

A common error committed by investors is that they focus only on the diversification among stocks traded on the major U.S. stock exchanges. But there are many other financial and non-financial instruments that can aid investors in diversifying their risk. Because these instruments are often less correlated with an investor's portfolio than domestic U.S. stocks, these assets can be especially valuable in reducing the portfolio risk. Among possible investment assets are:

- Savings accounts.
- Bonds.
- Commodities (such as gold).
- Other futures (such as agricultural commodities).
- Art.
- Real estate.
- Mortgage and corporate bonds.
- Labor income.
- International stocks.
- Hedge funds.
- Venture and private equity funds.
- Vulture and bankruptcy funds.

In addition, if you are a smart investor, you would not only consider the diversification within your stock portfolio, but across your entire wealth. Your wealth would include your house, your education, your job, etc. Many of these alternative investments could also have low covariation with your overall wealth.

Anecdote: Portfolios of Finance Professors

Many finance professors invest their own money into passively managed, low-cost mutual funds, often **Index Funds**, which buy-and-hold a wide cross-section of assets and avoid active trading. They also require minimal investment selection abilities by their managers, and usually incur minimal trading costs. Vanguard funds are particularly popular, because **Vanguard** is not only the largest mutual fund provider—though neck-in-neck with **Fidelity**—but it also does not even seek to earn a profit. It is a "mutual" mutual fund, owned by the investors in the funds themselves.

SIDE NOTE

Some of the above mentioned asset categories may be better held in modest amounts. Furthermore, some of these areas do not resemble the highly liquid, fair, and efficient financial markets that U.S. stock investors are used to. Instead, some are rife with outright scams. Therefore, it might be wise to hold such assets through sophisticated and dedicated professional investors (mutual funds), who have the appropriate expertise.

Solve Now!

Q 11.5 In Table 11.1, Exxon had the lowest standard deviation among the Dow-30 stocks. Why not just purchase Exxon by itself?

Q 11.6 How much does diversifying over all 500 stocks in the S&P500 help in terms of risk reduction relative to investing in the 30 stocks of the Dow Jones-30?

Q 11.7 Why do mutual funds exist?

Q 11.8 Should you just own U.S. stocks?

Q 11.9 Does the true value-weighted market portfolio just contain stocks?

11·5 Diversification Over Time

Explaining the Figure Illustrating Time Diversification.

Many investors think of diversification across securities within a portfolio, but do not realize that diversification can also work over time—although the sidenote below explains why academics are divided on this issue. Figure 11.4 illustrates this point by showing the risk and reward if you had invested in the S&P 500 from 1990–2002 for x consecutive trading days. The left graph shows your average daily rate of return; the right side your total compounded rate of return. For example, the right figure shows that if you had held an S&P500 portfolio during a random 25 day period (about a month), you would have earned a little less than 1%, but with a risk (standard deviation) of about 5%. Graph (a) quotes this in average daily terms: over 25 days, a 1% mean and 5% risk was an average reward of about 0.04%/day with a risk of about 0.2%/day.

Time Diversification at Work!

Note from graph (b) how the risk-reward trade-off changes with time. Over an investment horizon of a full year (255 trading days), you would have expected to earn about 10% with a risk of about 15%. The risk would have been 1.5 times the reward. In contrast, over an investment horizon of one day, you would have expected to earn about 0.04% with a risk of about 1.1%. Your risk would have been about 30 times your reward! If you stare at graph (b), you should notice that the reward goes up a little more than linearly (the compounding effect!), while the risk goes up like a parabola, i.e., a square root function. Indeed, this is the case, and the rest of this section shows why.

Over periods shorter than a few years, the ordinary portfolio return is roughly the sum of time period returns.

Recall that a portfolio that earns $\tilde{r}_{t=1}$ in period 1, $\tilde{r}_{t=2}$ in period 2, and so on until period T ($\tilde{r}_{t=T}$), will earn an overall rate of return of

$$\tilde{r}_{t=0,t=T} = (1+\tilde{r}_{t=0,1}) \cdot (1+\tilde{r}_{t=1,2}) \cdot \vdots \cdot (1+\tilde{r}_{t=T-1,T}) \quad -1$$

$$\approx \tilde{r}_{t=0,1} + \tilde{r}_{t=1,2} + \cdots + \tilde{r}_{t=T-1,T} + \text{many multiplicative } \tilde{r} \text{ terms}$$

The multiplicative terms reflect the power of compounding—which clearly matters over many years—but perhaps less so over periods of just a few months or years. After all, even monthly returns may typically be only on the order of 1%, so the multiplicative term would be on the order of $1\% \cdot 1\% = 0.01\%$.

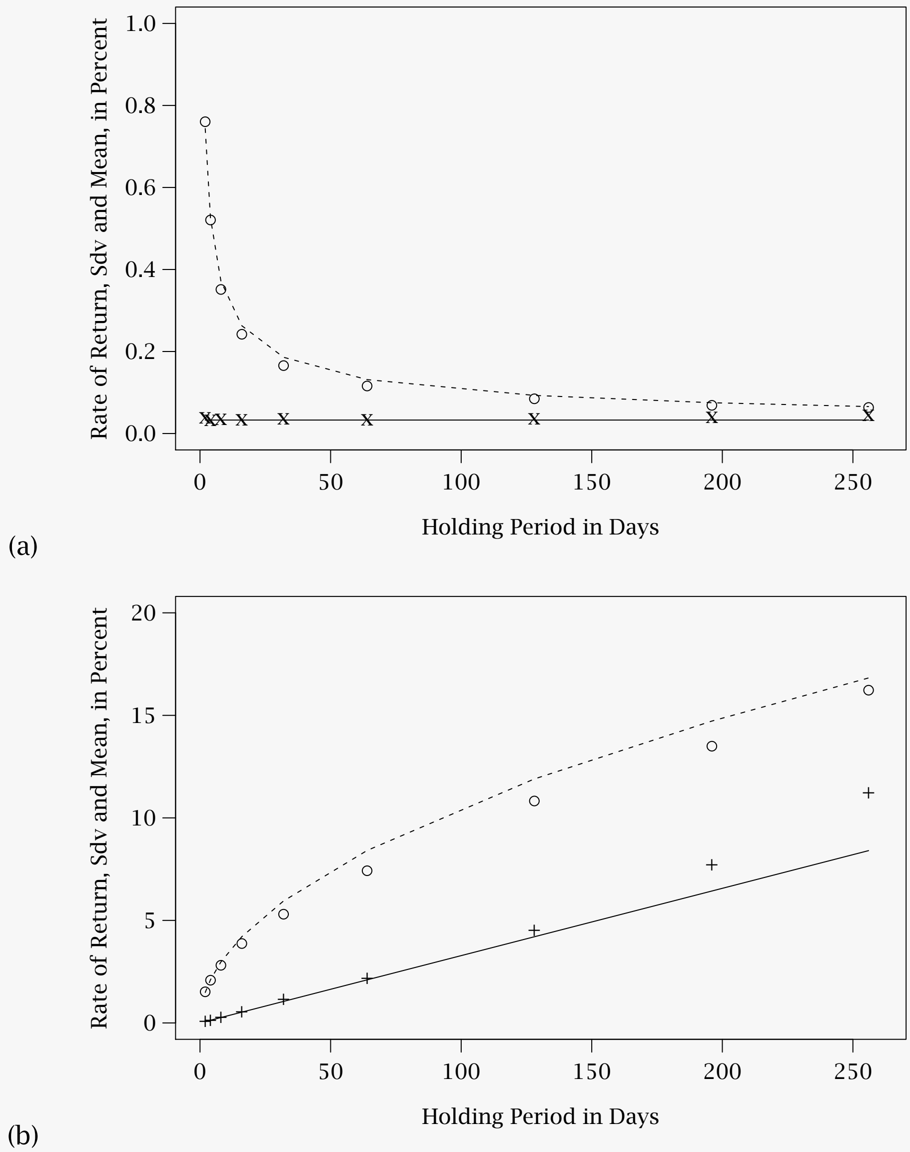
Stock returns have to be about independent across time-periods to avoid great money-making opportunities. This determines the riskiness of portfolios over multiple periods.

Now, if you forget about the small multiplicative terms, you already know that if you expect to earn 1% per period, then over x periods you expect to earn x%:

$$\mathcal{E}(\tilde{r}_{t=0,t=T}) = T \cdot \mathcal{E}(\tilde{r}_t)$$

For the variance, let us assume that the return variance in each time period is about the same and can just be called $Sdv(\tilde{r}_t)$. Further assume that the covariance terms among returns in different times (e.g., $\tilde{r}_{t=0,1}$ and $\tilde{r}_{t=1,2}$) are about zero. After all, if they were not, this would mean that you could predict future returns with past returns. Suffice it to say that this sort of

Figure 11.4: Average Risk and Return over Time in the S&P500, 1990–2002



The x-axis is the investment horizon, i.e., the number of consecutive investment days. The two top series (lines and dots) are the standard deviation, the two bottom series are the means. Lines are the theoretical values computed with the formulas below; circles and pluses are from simulated actual investment strategies.

prediction is not an easy task—if it were, you would quickly become rich! Put this all together and use the variance formula 11.3:

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_{t=0,T}) &= \mathcal{V}ar\,(\tilde{r}_{t=0,1} + \tilde{r}_{t=1,2} + \cdots + \tilde{r}_{t=T-1,T}) \\ &\quad + \text{ many covariance terms, all about zero} \qquad (11.3) \\ &\approx T \cdot \mathcal{V}ar(\tilde{r}_t) \end{aligned}$$

It follows that

$$Sdv(\tilde{r}_{t=0,T}) \approx \sqrt{T} \cdot Sdv(\tilde{r}_t) \qquad (11.4)$$

But this is just the relationship in the graph: The risk increases with the square root of time!

Annualization

Formula 11.4 is commonly used to "annualize" portfolio risk. For example, if an investment strategy has a monthly risk of 5% (i.e., the standard deviation of its rate of return), and the question is what kind of risk such an investment strategy would have per annum, you can compute the implied annual standard deviation to be about $\sqrt{12} \cdot 5\% \approx 17\%$. Conversely, if the five-year variance is 39%, then the annualized variance is 17%, because $\sqrt{5} \cdot 17\% \approx 39\%$.

IMPORTANT: A quick and dirty (and common) method to annualize portfolio risk (the standard deviation of the rate of return) is to multiply the single-period rate of return by the square root of the number of periods. For example, the annual portfolio risk is about $\sqrt{12}$ or 3.5 times as high the monthly risk.
The most common method, on Wall Street and in academia, is to compute risk and reward from monthly rates of return, but to report their annualized values.

The Sharpe-ratio is a measure of risk-reward performance.

The most commonly used measure of portfolio performance is the **Sharpe-ratio**, named after Nobel Prize Winner William Sharpe. It is the expected rate of return of a portfolio *above and beyond* the risk-free rate of return (r_F), divided by the standard deviation of the rate of the return,

$$\text{Sharpe Ratio} = \frac{\mathcal{E}(\tilde{r}_P - r_F)}{Sdv(\tilde{r}_P - r_F)} = \frac{\mathcal{E}(\tilde{r}_P) - r_F}{Sdv(\tilde{r}_P)}$$

Be aware that the Sharpe-ratio depends on the time interval that is used to measure returns. In the example with 1% monthly mean and 5% monthly standard deviation, if the risk-free rate were 6% per year, the Sharpe-Ratio of the portfolio would be about $^{(1\%\text{-}0.5\%)}/_{5\%} = 0.1$ if measured over a one month time interval; $^{(12\%\text{-}6\%)}/_{17\%} = 0.35$ if measured over a year; and $^{(60\%\text{-}30\%)}/_{39\%} = 0.77$ if measured over five years. Although the Sharpe ratio makes it clear how mean and standard deviation change with different time horizons, it is just a measure—it does not mean that you are better off if you hold stocks over longer time horizons. More importantly, you should be warned: although the Sharpe ratio is intuitively appealing and although it is in widespread use, it has the near-fatal flaw that it can easily be manipulated. You will find this out for yourself in Question 11.16.

SIDE NOTE

The overall division of assets between stocks and bonds is often called **asset allocation**. Many practitioners suggest that you should put more of your money into risky stocks when you are young. Over the very long run—and young people have naturally longer investment horizons—the expected rate of return vs. risk relationship looks more favorable than it does over shorter investment horizons. After all, the mean goes up with time, while the risk goes up only with the square-root of time. This means that the *average* rate of return is less risky when you are young than when you are old.

But academics are divided on this advice. Some point out that the portfolio rate of return is the product of the individual returns:

$$\tilde{r}_{t=0,T} \;=\; (1+\tilde{r}_{0,1})\cdot(1+\tilde{r}_{1,2})\cdot\cdots\cdot(1+\tilde{r}_{T-1,T}) \;-1$$

It should not matter whether you choose risky stocks over safer bonds at time $t = 0$ (when you are young) or at time $t = T$ (when you are old). That is, even though it is true that the risk of the *average* rate of return declines over longer horizon, you should not be interested in the *average* rate of return, but in the *total* rate of return. This argument suggests that your time horizon should not matter to your asset allocation.

Even more sophisticated arguments take into account that you can adjust better (e.g., by working harder) when you are young if you experience a bad portfolio return; or that the stock market rate of return may be **mean reverting**. That is, the market rate of return may be negatively correlated with itself over very long time periods, in which case the long-run risk could be a little lower than the short-run risk.

Solve Now!

Q 11.10 What is a reasonable assumption for stock return correlations across different time periods?

Q 11.11 Table 10.4 on Page 228 shows that the S&P500 has an annual standard deviation of about 20% per year. As of November 2002, the S&P500 stood at a level of about 900. What would you expect the daily standard deviation of the S&P500 index to be? Assume that there are about 255 trading days in a year.

Q 11.12 Assume you have a portfolio that seems to have a monthly standard deviation of 5%. What would you expect its annual standard deviation to be?

Q 11.13 If the risk-free rate over the 1997-2002 sample period was about 3% per annum, what was the monthly and what was the annual Sharpe-ratio of the Dow-Jones 30 index?

Q 11.14 Assume you have a portfolio that seems to have had a daily standard deviation of 1%. What would you expect its annual standard deviation to be? Assume there are 255 trading days in a year.

Q 11.15 The S&P500 was quoted in the first two weeks of June 2003 as

06/02/2003	967.00	06/06/2003	987.7606/12/2003	998.51
06/03/2003	971.56	06/09/2003	975.9306/13/2003	988.61
06/04/2003	986.24	06/10/2003	984.84	
06/05/2003	990.14	06/11/2003	997.48	

Compute the mean and standard deviation of annual returns of a portfolio that would have mimicked the S&P500. Based on these returns, what would you expect to be the risk and reward if you held the S&P500 for one year? Assuming a risk-free rate of return of 3%/annum, what would be the proper estimate for a Sharpe-ratio of daily, monthly, and annual returns? Is this consistent with the statistics in Table 10.4? Can you speculate why?

Q 11.16 Consider an investment strategy that has returned the following four rates of return: +5%, +10%, +5%, +20%. These are quoted above the risk-free rate (or equivalently assume the risk-free rate is 0%.)

(a) What was its Sharpe-ratio?

(b) Throw away 5% of the rate of return *in the final period only*. That is, if you had \$200 and you had ended up with \$240 (20%), you would now be throwing away \$10. You would end up with \$230 for a remaining rate of return of 15% only. (How easy is it to throw away money?) What is the Sharpe-ratio of this revised strategy?

(c) Which is the better investment strategy?

11·6 Summary

The chapter covered the following major points:

- Diversification—investment in different assets—reduces the overall risk (standard deviation).
- Diversification works better when assets are uncorrelated.
- The beta of a new asset (with respect to the existing portfolio) is a good measure of the marginal contribution of the new asset to the risk of the overall portfolio.
- Mutual funds invest in many assets, and thereby reduce retail investors' cost of diversification. However, they charge fees for this service.
- Diversification works over time, too. The portfolio reward (expected rate of return) grows roughly linearly over time, but the portfolio risk (standard deviation) grows roughly with the square-root of time. That is, if the expected rate of return is 1% per month, then the T month expected rate of return is approximately $T \cdot 1\%$. If the standard deviation of the rate of rate of return is 10% per month, then the T month standard deviation is approximately $\sqrt{T} \cdot 10\%$. The latter formula is often used to "annualize" risk.
- The Sharpe-ratio is the most common measure of investment strategy performance—although an awful one. One relatively minor problem is that it depends on the investment horizon on which it is quoted. A Sharpe-ratio based on annual returns is typically about $\sqrt{T}$ higher than a Sharpe-ratio based on monthly returns.

[No keyterm list for diversification-g.]

End of Chapter Problems

16 "Solve Now" Answers

1. The mean is 12%, the standard deviation is $30\%/\sqrt{N}$.
2. The mean is 12%. The variance now still has N variance terms, but $N \cdot (N-1)$ terms that are each $1/N \cdot 1/N \cdot 0.001$. Thus, the variance is now

$$\mathcal{V}ar(\tilde{r}_P) = N \cdot (1/N^2 \cdot (30\%)^2) + N \cdot (N-1) \cdot 1/N \cdot 1/N \cdot (0.0025)$$

Therefore, for 2 stocks, the variance is $1/2 \cdot .09 + 2 \cdot 1 \cdot (1/4 \cdot 0.0025) = 0.045 + 1/2 \cdot 0.0025 = 0.04625$. $Sdv = 21.5\%$. This is a little higher than the 21.2% from the previous question. For many stocks, $N \to \infty$ is

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_P) &= N \cdot (1/N^2 \cdot (30\%)^2) + N \cdot (N-1) \cdot 1/N \cdot 1/N \cdot (0.0025) \\ &\approx N \cdot (1/N^2 \cdot (30\%)^2) + 0.0025 \\ &= 1/N \cdot (.09) + 0.0025 \\ \lim_{N\to\infty} Sdv(\tilde{r}_P) &= \sqrt{0.0025} = 5\% \end{aligned}$$

3.

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= \mathcal{V}ar(\,1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2\,) \\
&= 1/2^2 \cdot \mathcal{V}ar(\,\tilde{r}_1\,) + 1/2^2 \cdot \mathcal{V}ar(\,\tilde{r}_2\,) + 2 \cdot 1/2 \cdot 1/2 \cdot Cov(\,\tilde{r}_1, \tilde{r}_2\,) \\
&= 1/4 \cdot 0.01 + 1/4 \cdot 0.01 + 2 \cdot 1/2 \cdot 1/2 \cdot (-0.01) \\
&= 1/2 \cdot 0.01 - 1/2 \cdot 0.01 \quad = \quad 0.00 \\
Sdv(\tilde{r}_P) &= \sqrt{\mathcal{V}ar(\tilde{r}_P)} \;=\; 0\% \quad .
\end{aligned}$$

Therefore, with perfect negative correlation and equal investment weights, diversification works perfect!

4.

(a) The covariance is the correlation multiplied by the two standard deviations. For the Q portfolio,

$$\begin{aligned}
Cov(\tilde{r}_Q, \tilde{r}_P) &= Correlation(\tilde{r}_Q, \tilde{r}_P) \cdot Sdv(\tilde{r}_Q) \cdot Sdv(\tilde{r}_P) \\
&= 20\% \cdot 20\% \cdot 50\% \quad = \quad 0.02
\end{aligned}$$

(b)

$$\beta_{Q,P} = \frac{Cov(\tilde{r}_Q, \tilde{r}_P)}{\mathcal{V}ar(\tilde{r}_P)} = \frac{0.02}{0.04} = 0.5$$

(c) Therefore, the new portfolio with weights of 0.5 each is

$$\begin{aligned}
\tilde{r} &= 0.5 \cdot \tilde{r}_P + 0.5 \cdot \tilde{r}_Q \\
\mathcal{V}ar(\tilde{r}) &= w_P \cdot \mathcal{V}ar(\tilde{r}_P) + w_Q \cdot \mathcal{V}ar(\tilde{r}_Q) + 2 \cdot w_Q \cdot w_P Cov(\tilde{r}_Q, \tilde{r}_P) \\
&= 0.5^2 \cdot (20\%)^2 + 0.5^2 \cdot (50\%)^2 + 2 \cdot 0.5 \cdot 0.5 \cdot 0.02 \\
&= 0.01 + 0.0625 + 0.01 \;=\; 0.0825 \\
Sdv(\tilde{r}) &= 28.7\%
\end{aligned}$$

(d) For the N portfolio,

$$\begin{aligned}
Cov(\tilde{r}_N, \tilde{r}_P) &= Correlation(\tilde{r}_N, \tilde{r}_P) \cdot Sdv(\tilde{r}_N) \cdot Sdv(\tilde{r}_P) \\
&= 20\% \cdot 20\% \cdot 500\% \quad = \quad 0.2
\end{aligned}$$

(e)

$$\beta_{\tilde{r}_N, \tilde{r}_P} = \frac{Cov(\tilde{r}_N, \tilde{r}_P)}{\mathcal{V}ar(\tilde{r}_P)} = \frac{0.2}{0.04} = 5$$

(f) Therefore, the new portfolio with weights of 0.5 each is

$$\begin{aligned}
\tilde{r} &= 0.5 \cdot \tilde{r}_P + 0.5 \cdot \tilde{r}_N \\
\mathcal{V}ar(\tilde{r}) &= w_P \cdot \mathcal{V}ar(\tilde{r}_P) + w_Q \cdot \mathcal{V}ar(\tilde{r}_Q) + 2 \cdot w_Q \cdot w_P Cov(\tilde{r}_Q, \tilde{r}_P) \\
&= 0.5^2 \cdot (20\%)^2 + 0.5^2 \cdot (500\%)^2 + 2 \cdot 0.5 \cdot 0.5 \cdot 0.2 \\
&= 0.01 + 6.25 + 0.1 \;=\; 6.36 \\
Sdv(\tilde{r}) &= 252.2\%
\end{aligned}$$

(g) Even though N has an equal correlation with portfolio P, its diversification aid is overwhelmed by its scale: N is just a lot riskier than Q. Thus, the portfolio with N is definitely more risky than the portfolio with Q. This lesser diversification effect is reflected in the (higher) covariance and the (higher) beta with our portfolio P, but not in the (equal) correlation.

5. This is not a question you might necessarily know how to answer. However, it should get you to think. Exxon is indeed a low-risk stock. However, it is also a low mean stock. You would have done a little better purchasing the diversified Dow-30 index (remember: indexes do not count dividends!). However, it is reasonably likely that in the future, the diversified Dow-Jones 30 index will have lower risk than Exxon. More than likely, Exxon was just lucky over the sample period. For example, oil prices were relatively stable.

6. There is almost no difference in risk between these two indexes. To reduce risk further, you should instead invest in asset classes that are different from large corporate equities.

7. They allow investors with limited amounts of money to own large numbers of securities, without incurring large transaction costs.

8. No. You should hold all sorts of other assets that do not correlate too highly with your existing portfolio (wealth).
9. No. There are many alternative asset classes, such as commodities, art, and real estate, etc., which are part of the market portfolio.

10. That they are zero. If they were not zero, it would mean that you could use past returns to predict future returns. Presumably, this would allow you to earn extra returns.
11. The standard deviation is likely to be

$$\begin{aligned} Sdv_{\text{annual}}(\tilde{r}) &\approx \sqrt{T} \cdot Sdv_{\text{daily}}(\tilde{r}) \\ 20\% &\approx \sqrt{255} \cdot Sdv_{\text{daily}}(\tilde{r}) \\ \Rightarrow \quad Sdv_{\text{daily}}(\tilde{r}) &\approx 1.25\% \end{aligned}$$

This corresponds to a typical daily movement of about 11 points. In statistical terms, if the S&P500 is about normally distributed, about $^2/_3$ of all days, it should move up or down no more than 11 points. In about $^9/_{10}$ of all days, it should move up or down no more than 22 points.
12. $\sqrt{12} \cdot 5\% \approx 17.3\%$.
13. For the annual Sharpe-ratio, you can use the numbers in the Table 11.1:

$$\text{Annual Sharpe Ratio} = \frac{6.3\% - 3.0\%}{18.7\%} \approx 0.18$$

For the monthly Sharpe-ratio, you need to de-annualize the mean and standard deviation. The excess mean is $3.3\%/12 \approx 0.275\%$. The monthly standard deviation is about $18.7\%/\sqrt{12} \approx 5.4\%$. Therefore, the monthly

$$\text{Monthly Sharpe Ratio} = \frac{0.275\%}{5.4\%} \approx 0.05$$

14. $\sqrt{255} \cdot 1\% \approx 16.0\%$.
15. Over 10 real days, you would have earned a compound rate of return of $988.61/967 - 1 = 2.2\%$. With 36.5 such 10-day periods over the year, the compound annual return would have been 124%. As to the daily rates of return, they were 0.47%, 1.51%, 0.40%, –0.24%, –1.20%, 0.92%, 1.28%, 1.03%, –1.00%. The simple arithmetic average rate of return was an even higher 0.35%/day (3.5% over the 10 days). Clearly, this was a great ten days, not likely to repeat. You should not trust these 10 day means. The standard deviation is 0.97%/day. This indicates an annualized standard deviation of about $\sqrt{365} \cdot 0.97\% \approx 18.5\%$. (Over 10 days, the estimated standard deviation is 3.07%/(**10 days**).) As is fairly common, the annualized standard deviation is fairly reasonable. Finally a risk-free rate of return of 3% per annum is less than 0.01%./day. The Sharpe-ratio would therefore be approximately $(0.35\% - 0.01\%)/0.97\% \approx 0.35$. Quoted in annual terms, the Sharpe-ratio would be approximately $(0.35\% - 0.01\%) \cdot 365/(0.97\% \cdot \sqrt{365}) = \sqrt{365} \cdot 0.35 \approx 6.7$. Quoted in monthly terms, it would be $\sqrt{30} \cdot 0.35 \approx 1.9$.
16.
    (a) The average rate of return is 10%. The variance is 37.5%%. The standard deviation is 6.12%. Therefore, its Sharpe-ratio is 1.63.
    (b) The new strategy has rates of return of +5%, +10%, +5%, +15%. It is very easy to accomplish this—give me the money. The average rate of return has declined to 8.75%. The standard deviation has declined to 4.146%. Therefore, the Sharpe ratio is 2.11.
    (c) Obviously, the second strategy of throwing money away is terrible. The fact that the Sharpe-ratio comes out higher tells us that the Sharpe-ratio is an awful measure of portfolio performance. And, yes, the Sharpe-ratio is indeed the most common fund performance measure in practical use. Sharpe ratio manipulation is particularly profitable for portfolio managers whose returns until October or November were positive. In this case, managers who want to maximize Sharpe ratios should try to avoid really high positive rates of returns. If it "happens," they can always bring down the return, e.g., by paying themselves more money, or by buying illiquid securities and then marking them down to less than their values.

All answers should be treated as suspect. They have only been sketched and have not been checked.